Chaos and Logistic Map

モデリングとシミュレーション特論

2019年度

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Chaos

- Henri Poincaré
  - complex trajectories for 3-body problems (1880's)
- Edward Lorenz
  - difficulties in weather forecasts (1960's)
  - small initial differences expands.
- Turbulences

Logistic Map

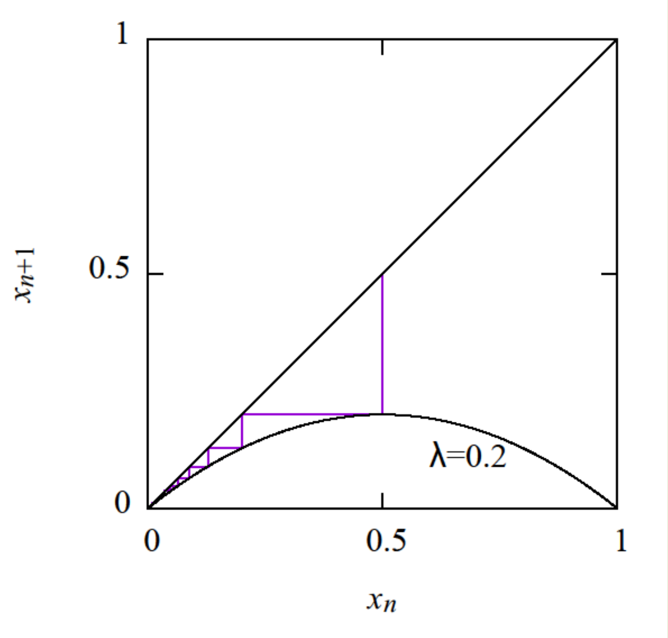
- Simplest model of chaotic behavior
- A species which has off-springs
  - If the number of individuals small, the number of off-springs will increase proportionally.
  - If large, the number of off-springs will degrease.

$$x_{n+1} = f_\lambda(x_n), \quad f_\lambda(x) = 4\lambda x(1-x)$$

$$x_i \in [0,1], \quad \lambda \in [0,1]$$

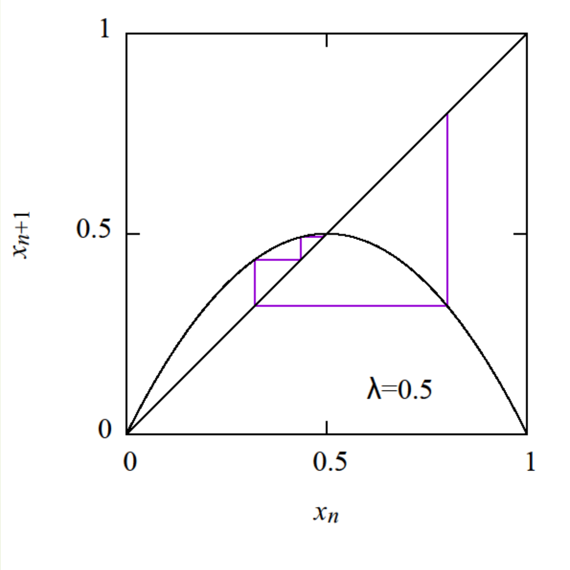
fixed points for small λ

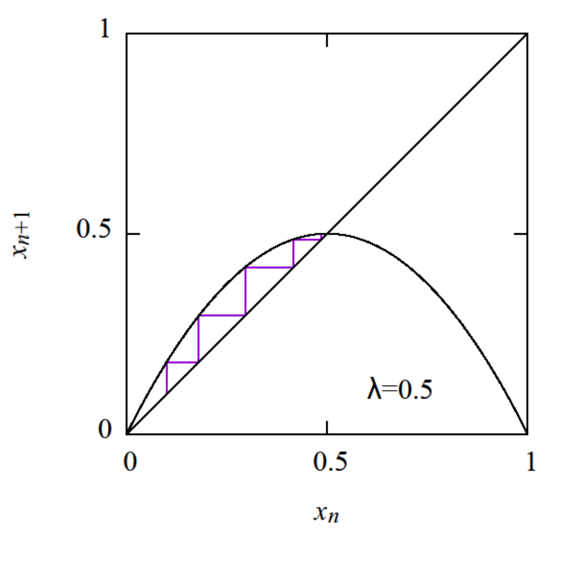
- fixed points are solutions of $x = f_\lambda(x)$
- $\lambda < 1/4$
  - only one fixed point $x = 0$



fixed points for small λ

- $1/4 < \lambda < 3/4$
  - two fixed points $x = 0, (4\lambda - 1)/(4\lambda)$
  - Trajectories do not go to $x = 0$.





stability of fixed points

- A point $x_0 = x_f + \delta$ near a fixed point x_f

$$x_1 = f_\lambda\left(x_f + \delta\right) = f_\lambda\left(x_f\right) + \delta \left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_f} + O\left(\delta^2\right)$$

- Stable : $|\mathrm{d}f_\lambda/\mathrm{d}x| < 1$
- Unstable : $|\mathrm{d}f_\lambda/\mathrm{d}x| > 1$

Stability of $x_f = 0$

$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=0} = 4\lambda\left(1-2x\right)\Big|_{x=0} = 4\lambda$$

- stable for $\lambda < 1/4$
- unstable for $\lambda > 1/4$

Stability of $x_f = (4\lambda - 1)/(4\lambda)$

$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_f} = 4\lambda\left(1-2x\right)\Big|_{x=x_f} = 2-4\lambda$$

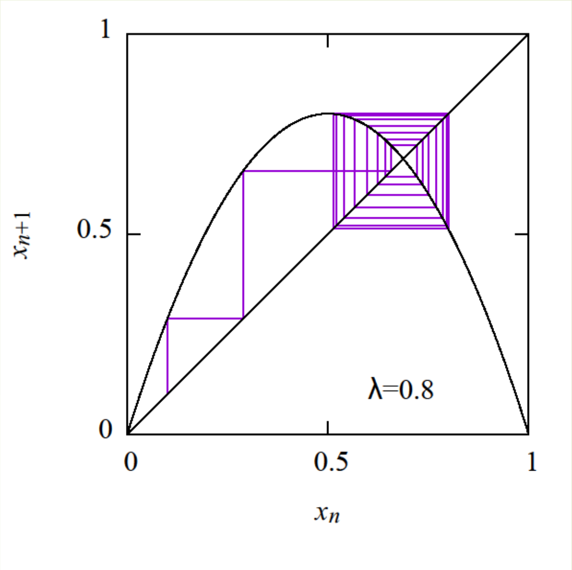
- $|\mathrm{d}f_\lambda/\mathrm{d}x| = 1$ at $\lambda = 1/4$
- $|\mathrm{d}f_\lambda/\mathrm{d}x| = -1$ at $\lambda = 3/4$
- stable for $1/4 < \lambda < 3/4$

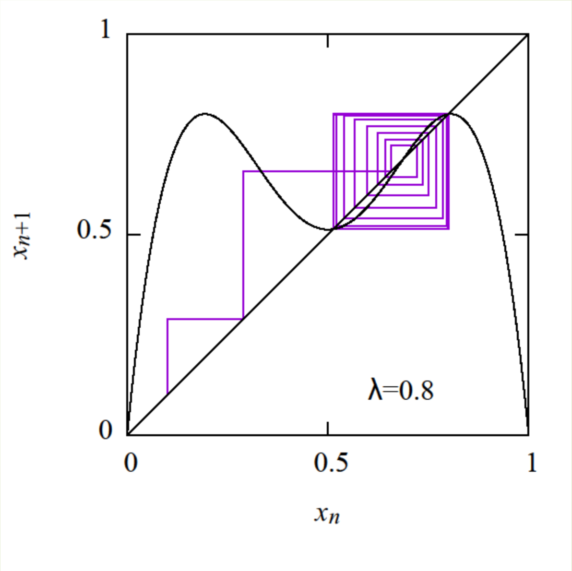
Period Doubling

- $\lambda = 0.8$
- Period-2 orbit

$$x_{\pm} = f_{\lambda}\left(x_{\mp}\right)$$

$$x_{\pm} = \frac{1}{8\lambda}\left[4\lambda + 1 \pm \sqrt{\left(4\lambda+1\right)\left(4\lambda-3\right)}\right]$$



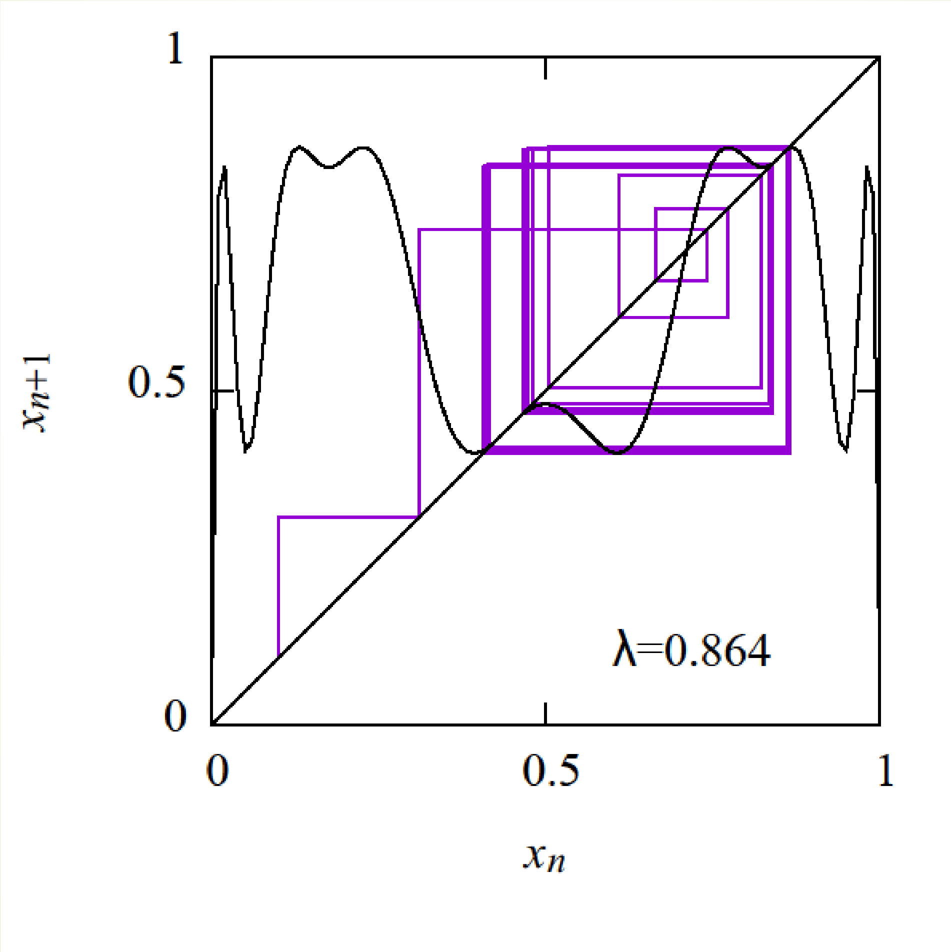


Stability of period-2 trajectories

$$f_\lambda^{[n+1]}(x) = f_\lambda\left(f_\lambda^{[n]}(x)\right),$$
$$f_\lambda^{[1]}(x) = f_\lambda(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{2}(x)\Big|_{x=x_\pm} = 1-(4\lambda+1)(4\lambda-3)$$

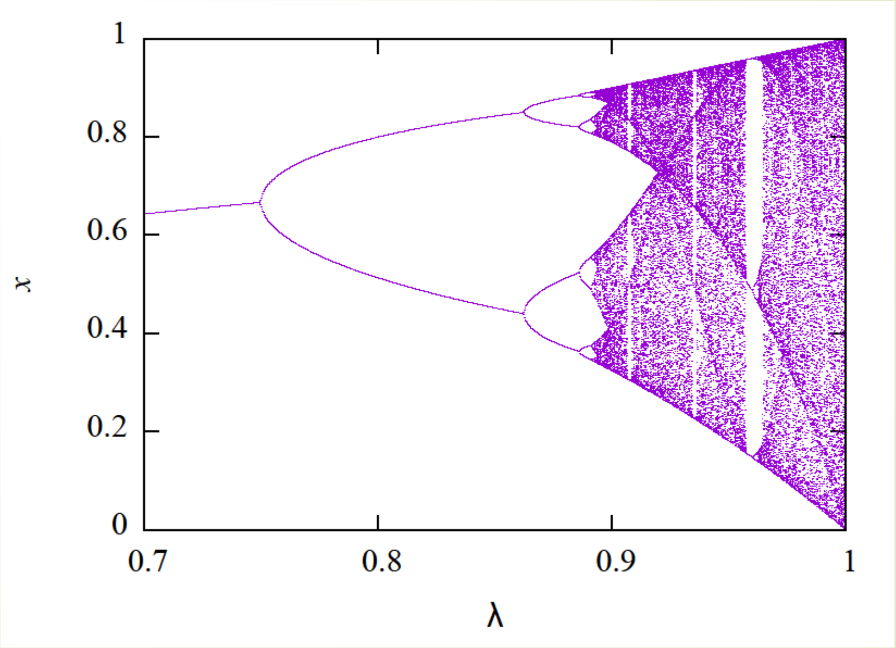
- Where is the next instability?
  - $\lambda = \left(1+\sqrt{6}\right)/4 \approx 0.8624$



1
0.5
0
0
0.5
1
x_{n+1}
x_n
λ=0.864

Period doubling to chaos

- Trajectories are doubled by increasing λ
- Period becomes infinite at $\lambda \approx 0.893$



Sample Programs

- https://github.com/modeling-and-simulation-mc-saga/Logistic
- model/Logistic.java
  - Logistic map
  - setting λ
  - update() method

- analysis/PrintOrbit.java
  - show orbits in (x_n, x_{n+1})-plane
  - show Logistic map : $f_{\lambda}^{[n]}(x)$

Direct output to PDF

- utils/Gnuplot.java
  - open gnuplot as a process
  - open outputstream of the process
  - write gnuplot commands to the stream