Differential Equations : external forces

モデル化とシミュレーション特論
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Numerical methods for Differential Equations

- Differential Equations
  t : independent variables, $\vec{y}$: dependent variables

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{y} = \vec{f}(t, \vec{y}) \tag{1}$$

- Euler method: simplest numerical method: advance t with h.

$$(t_n, \vec{y}_n) \to (t_{n+1} = t_n + h, \vec{y}_{n+1}) \tag{2}$$

$$\vec{y}_{n+1} = \vec{y}_n + h\vec{f}(t_n, \vec{y}_n) \tag{3}$$

Runge-Kutta method

$$\vec{k}_1 = h\vec{f}\,(t_n, \vec{y}_n)$$
$$\vec{k}_2 = h\vec{f}\left(t_n + \frac{h}{2}, \vec{y}_n + \frac{\vec{k}_1}{2}\right)$$
$$\vec{k}_3 = h\vec{f}\left(t_n + \frac{h}{2}, \vec{y}_n + \frac{\vec{k}_2}{2}\right)$$
$$\vec{k}_4 = h\vec{f}\left(t_n + h, \vec{y}_n + \vec{k}_3\right)$$

$$\vec{y}_{n+1} = \vec{y}_n + \frac{1}{6}\left(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4\right) + O\left(h^5\right) \qquad (4)$$

Correct up to $O(h^4)$

Differential Equations with Java

- Runge-Kutta method
  Obtain values of dependent variables $\vec{y}(t+h)$ at $t+h$ from $\vec{y}(t)$ and $\mathrm{d}\vec{y}/\mathrm{d}t = \vec{f}(t, \vec{y})$
- Runge-Kutta method can be described as subprograms
  static method which does not affect the properties of the instance.
- Sample programs
  `https://github.com/modeling-and-simulation-mc-saga/DifferentialEquations`

Function as a method argument

- java does not have *pointers*
- Functions are passed to methods as an instance of interface
- However, how to create an instance of interface

Instances of Interfaces

- Anonymous class : Implements method `apply()` at the construction.

```
DoubleFunction<Double> function
    = new DoubleFunction<Double>(){
        @Override
        public Double apply(double v){
            return v*v;
        }
};
```

- lambda expression

```
DoubleFunction<Double> function = x -> x * x;
```

`myLib.RungeKutta`

- `DifferentialEquation.java`
  - an interface
  - Right hand side of differential equations
- `RungeKutta.java`
  - implement Runge-Kutta method
  - advance time with h
  - advance time with given steps

Harmonic Oscillators

- Harmonic Oscillators

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx \tag{5}$$

$$x(t) = A\cos\left(\omega t + \delta\right), \quad \omega^2 = \frac{k}{m} \tag{6}$$

- In a form of first-order differential equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v \tag{7}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{k}{m}x \tag{8}$$

Periodic External Force

- Interesting phenomena such as resonance appear under periodic external forces

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 + \frac{1}{m} F(t) \tag{9}$$

$$F(t) = f \cos\left(\gamma t + \beta\right) \tag{10}$$

Homogeneous and Inhomogeneous equations

- Homogeneous equations : the order in both size is equal

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = G(t)x \tag{11}$$

- general solutions

$$x(t) = Ax_+(t) + Bx_-(t) \tag{12}$$

Homogeneous and Inhomogeneous equations

- Homogeneous equations plus an inhomogeneous term $F(t)$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = G(t)x + F(t) \tag{13}$$

- special solution $x_0(t)$

$$\frac{\mathrm{d}^2 x_0}{\mathrm{d}t^2} = G(t)x_0 + F(t) \tag{14}$$

- General solutions for inhomogeneous equations

$$x(t) = Ax_+(t) + Bx_-(t) + x_0(t) \tag{15}$$

Special Solutions

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\omega^2 + \frac{1}{m}F(t) \tag{16}$$

$$F(t) = f\cos(\gamma t + \beta) \tag{17}$$

- Assume a form of the special solution as $x_0(t) = B\cos(\gamma t + \beta)$

$$-\gamma^2 B\cos(\gamma t + \beta) = -\omega^2 B\cos(\gamma t + \beta) + \frac{f}{m}\cos(\gamma t + \beta) \tag{18}$$

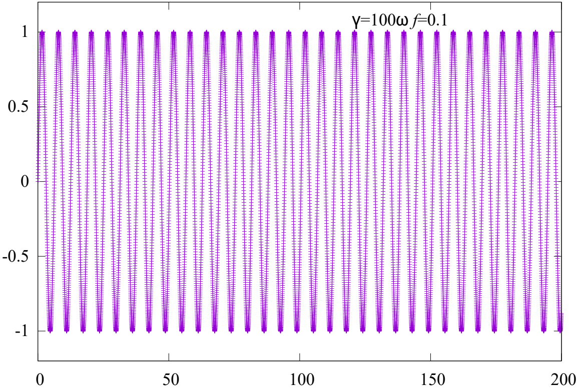
$$x_0(t) = \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta) \tag{19}$$

General Solutions

- general solutions for homogeneous equation plus special solution

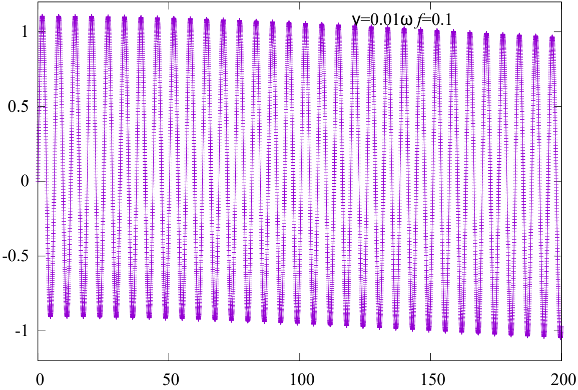
$$x(t) = A\cos(\omega t + \delta) + \frac{f}{m(\omega^2 - \gamma^2)}\cos(\gamma t + \beta) \tag{20}$$

Fast External Force : $\gamma \gg \omega$



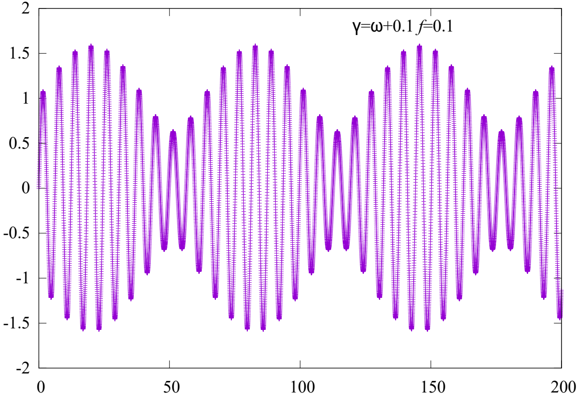
- the external force changes faster than one period of the oscillator
- the external force changes too fast to affect the oscillator

Slow External Force : $\gamma \ll \omega$

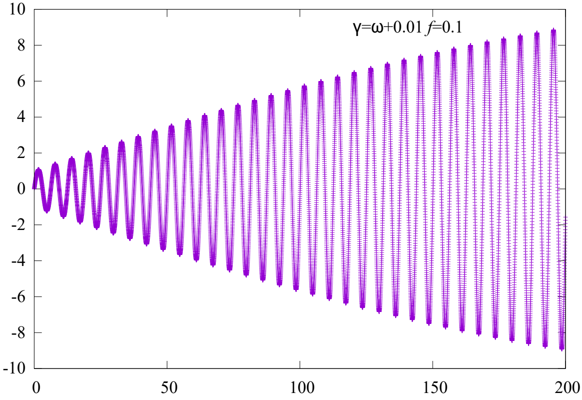


- Oscillation under slow external force
- oscillation itself are almost not affected

External Force with Close Frequency: resonance



External Force with Very Close Frequency: howling



- howling : the amplitude grows linearly with time

- $\gamma = \omega + \epsilon$, $\epsilon \ll 1$

$$\cos(\gamma t + \beta) - \cos(\omega t + \beta) = -t\epsilon \sin(\omega t + \beta) + O\left(\epsilon^2\right) \tag{21}$$

$$\frac{1}{\omega^2 - \gamma^2} = -\frac{1}{2\omega}\left(1 + O(\epsilon)\right) \tag{22}$$

- howling

$$x(t) = A' \cos\left(\omega t + \alpha'\right) + t\frac{f}{2m\omega}\sin(\omega t + \beta) + O(\epsilon) \tag{23}$$