Random Walk and Central Limiting Theorem

モデル化とシミュレーション特論
2021 年度前期
佐賀大学理工学研究科　只木進一

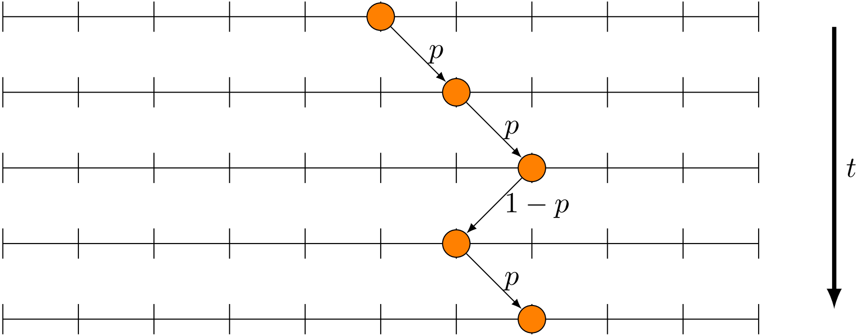
Stochastic Processes (確率過程)

- System evolves non-deterministically
  - evolve with probability
- random walks (酔歩)
  - Fundamental model of stochastic processes
  - one-dimensional lattice
  - At every step, move right with p and left with $1-p$

Sample Program

`https://github.com/modeling-and-simulation-mc-saga/RandomWalk`

Image of random walk



Theoretical Analysis

- Position x of a particle starting from $x = 0$
- At t, a particle position is x if the particle moves right $m = (t + x)/2$.
  - Note possible combination of left and right movements.
- binomial distribution

$$P(x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2} (1-p)^{(t-x)/2} \tag{1}$$

Generating Function

- Tool for evaluating moments such as average and deviation.
- Convert probability $P(x)$ for x to probability $Q(m)$ for m

$$P(x) : x = 2m - t, \ \ m \in [0, t] \tag{2}$$

$$Q(m) : m = \frac{t + x}{2}, \ \ m \in [0, t] \tag{3}$$

- Generating function

$$G(z) = \sum_{m=0}^{t} Q(m) z^m \tag{4}$$

General theory of Generating Functions and Moments

$$G(1) = \sum_{m=0}^{t} Q(m) = 1 \tag{5}$$

$$G'(z) = \sum_{m=1}^{t} mQ(m)z^{m-1} = \sum_{m=0}^{t} mQ(m)z^{m-1} \tag{6}$$

$$G'(1) = \sum_{m=0}^{t} mQ(m) = \langle x \rangle \tag{7}$$

$$G''(z) = \sum_{m=2}^{t} m(m-1)Q(m)z^{m-2} = \sum_{m=0}^{t} m(m-1)Q(m)z^{m-2} \tag{8}$$

$$G''(1) = \sum_{m=0}^{t} m(m-1)Q(m) = \langle m^2 \rangle - \langle m \rangle \tag{9}$$

Generating Function for Binomial Distribution

$$G(z) = \sum_{m=0}^{t} \binom{t}{m} p^m (1-p)^{t-m} z^m = (zp + 1 - p)^t \tag{10}$$

$$G(1) = 1 \tag{11}$$

$$G'(z) = tp\,(zp + 1 - p)^{t-1} \tag{12}$$

$$\langle m \rangle = tp \tag{13}$$

$$\langle x \rangle = \langle 2m - 1 \rangle = 2tp - t = t\,(2p - 1) \tag{14}$$

$$G''(z) = t\,(t-1)\,p^2\,(zp+1-p)^{t-2} \tag{15}$$

$$G''(1) = \left\langle m^2 \right\rangle - \langle m \rangle = t\,(t-1)\,p^2 \tag{16}$$

$$\begin{aligned}
\sigma^2 &= \left\langle x^2 \right\rangle - \langle x \rangle^2 = \left\langle 4m^2 - 4mt + t^2 \right\rangle - \langle x \rangle^2 \\
&= 4\left(\left\langle m^2 \right\rangle - \langle m \rangle\right) + 4\,\langle m \rangle\,(1-t) + t^2 - \langle x \rangle^2 \\
&= 4G''(1) + 4\,\langle m \rangle\,(1-t) + t^2 - \langle x \rangle^2 \\
&= 4tp^2\,(t-1) + 4tp\,(1-t) + t^2 - t^2\left(4p^2 - 4p + 1\right) \\
&= \boxed{4tp\,(1-p)}
\end{aligned} \tag{17}$$

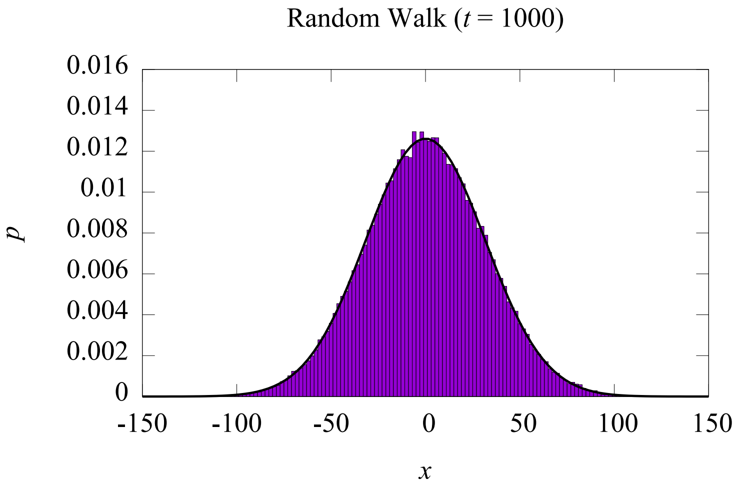
Approximated shape at the visinity of the mean

$$P(x) \propto \exp\left[-\frac{\left(x - \langle x \rangle\right)^2}{2\sigma^2}\right]$$

$$\langle x \rangle = t\left(2p - 1\right)$$

$$\sigma^2 = 4tp\left(1 - p\right)$$

- Normal distribution
- Is this a special for this random walk?


Random Walk (t = 1000)
0.016
0.014
0.012
0.01
0.008
0.006
0.004
0.002
0
p
-150
-100
-50
0
50
100
150
x

Random Walk in other viewpoints

- Sequence of random variables $\{X_i\}$ with $P(x)$

$$P(x) = \begin{cases} p & x = 1 \\ 1 - p & x = -1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

- Sum S_i of the random variables

$$S_0 = 0 \tag{19}$$

$$S_n = S_{n-1} + X_{n-1} = \sum_{k=0}^{n-1} X_k \tag{20}$$

- S_n: position of a walker at $t = n$
- Distribution of S_n is the distribution of walkers at $t = n$

Extension of one-dimensional Random Walks

- Sequence of random variables $\{X_k\}$ obeying probability density $f(x)$

$$S_0 = 0 \tag{21}$$

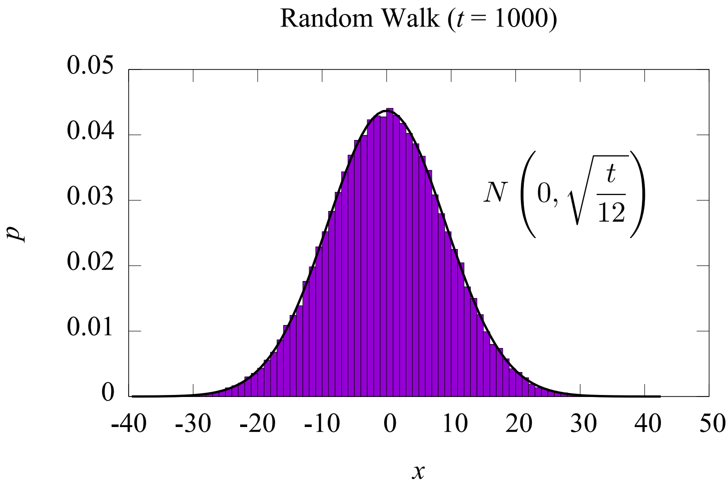
$$S_n = S_{n-1} + X_{n-1} = \sum_{k=0}^{n-1} X_k \tag{22}$$

- Note that S_i is continuous.

Example: uniform distribution

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

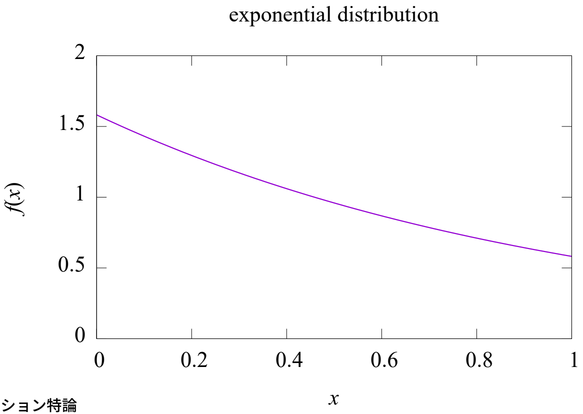
$$\begin{aligned} \langle x \rangle &= \int_{-1/2}^{1/2} x f(x) \mathrm{d}x = \int_{-1/2}^{1/2} x \mathrm{d}x = \left[\frac{1}{2} x^2 \right]_{-1/2}^{1/2} = 0 \\ \langle x^2 \rangle &= \int_{-1/2}^{1/2} x^2 f(x) \mathrm{d}x = \int_{-1/2}^{1/2} x^2 \mathrm{d}x = \left[\frac{1}{3} x^3 \right]_{-1/2}^{1/2} = \frac{1}{12} \\ \sigma^2 &= \frac{1}{12} \end{aligned} \tag{24}$$


Random Walk (t = 1000)
0.05
0.04
0.03
0.02
0.01
0
p
-40
-30
-20
-10
0
10
20
30
40
50
x
$N\left(0, \sqrt{\frac{t}{12}}\right)$

Example: Exponential distribution

$$f(x) = Ae^{-x}, \quad (0 \leq x < 1) \tag{25}$$

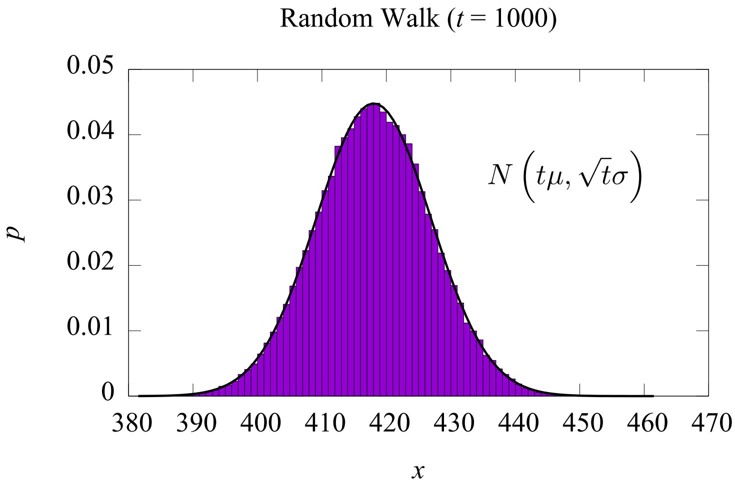
$$A = \frac{e}{e-1} \tag{26}$$



$$\begin{aligned}\langle x\rangle &= \int_0^1 Axe^{-x}\mathrm{d}x = \int_0^1 Ae^{-x}\mathrm{d}x - A\left[xe^{-x}\right]_0^1 \\ &= 1-\frac{e}{e-1}e^{-1} = \frac{e-2}{e-1} = 1-\frac{1}{e-1} \end{aligned} \tag{27}$$

$$\begin{aligned}\left\langle x^2\right\rangle &= \int_0^1 Ax^2e^{-x}\mathrm{d}x = 2\int_0^1 Axe^{-x}\mathrm{d}x - A\left[x^2e^{-x}\right]_0^1 \\ &= 2\frac{e-2}{e-1}-\frac{e}{e-1}e^{-1} = \frac{2e-5}{e-1} = 2-\frac{3}{e-1} \end{aligned} \tag{28}$$

$$\sigma^2 = \left\langle x^2\right\rangle - \langle x\rangle^2 = \frac{e^2-3e+1}{(e-1)^2} = 1-\frac{e}{(e-1)^2} \tag{29}$$



Random Walk (t = 1000)
0.05
0.04
0.03
0.02
0.01
0
p
$N\left(t\mu, \sqrt{t}\sigma\right)$
380 390 400 410 420 430 440 450 460 470
x

Central Limiting Theorem (中心極限定理)

- $\{X_k\}$: random variables obeying an identical distribution with the mean μ and deviation σ^2

$$S_n = \sum_{k=1}^{n} X_k \tag{30}$$

$$S_n^* = \frac{S_n - n\mu}{\sqrt{n}\sigma} \tag{31}$$

$$\lim_{n\to\infty} P(S_n^* \leq a^*) = \frac{1}{\sqrt{2}} \int_{-\infty}^{a*} \exp\left[-\frac{x^2}{2}\right] \mathrm{d}x \tag{32}$$

- In the limit $n \to \infty$, S_n^* obeys the standard normal distribution $N\left(0, 1\right)$

Probability Characteristic Function (特性関数)

- Continuous probability density $f(x)$ defined in $[a, b)$

$$G(t) = \int_a^b f(x)e^{itx}\mathrm{d}x \tag{33}$$

$$G(0) = \int_a^b f(x)\mathrm{d}x = 1 \tag{34}$$

$$G'(t) = \int_a^b ixf(x)e^{itx}\mathrm{d}x \tag{35}$$

$$G'(0) = i\int_a^b xf(x)\mathrm{d}x = i\langle x\rangle \tag{36}$$

$$G''(t) = -\int_a^b x^2 f(x)e^{itx}\mathrm{d}x \tag{37}$$

$$G''(0) = -\int_a^b x^2 f(x)\mathrm{d}x = -\langle x^2\rangle \tag{38}$$

$$\langle x \rangle = -iG'(0) \tag{39}$$

$$\left\langle x^2 \right\rangle = -G''(0) \tag{40}$$

$$\sigma^2 = -G''(0) + G'(0)^2 \tag{41}$$

Example: uniform

$$f(x) = \begin{cases} 1 & -\frac{1}{2} \leq x < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{42}$$

$$\begin{aligned} G(t) &= \int_{-1/2}^{1/2} e^{itx} \mathrm{d}x = \left[\frac{1}{it} e^{itx} \right]_{-1/2}^{1/2} \\ &= \frac{1}{it} \left(e^{it/2} - e^{-it/2} \right) = \frac{2 \sin\left(\frac{t}{2}\right)}{t} \\ &= \frac{2}{t} \left(\frac{t}{2} - \frac{1}{6} \left(\frac{t}{2} \right)^3 + \frac{1}{5!} \left(\frac{t}{2} \right)^5 + O\left(t^7\right) \right) \\ &= 1 - \frac{1}{24} t^2 + \frac{1}{5! 2^4} t^4 + O\left(t^6\right) \end{aligned} \tag{43}$$

$$G(0) = 1 \tag{44}$$

$$G'(t) = -\frac{1}{12}t + \frac{1}{5!2^3}t^3 + O\left(t^5\right) \tag{45}$$

$$G'(0) = 0 \tag{46}$$

$$G''(t) = -\frac{1}{12} + \frac{3}{5!2^3}t^2 + O\left(t^4\right) \tag{47}$$

$$G''(0) = -\frac{1}{12} \tag{48}$$

$$\langle x \rangle = 0 \tag{49}$$

$$\left\langle x^2 \right\rangle = \frac{1}{12} \tag{50}$$

$$\sigma^2 = \frac{1}{12} \tag{51}$$

Example: Exponential

$$f(x) = Ae^{-x}, \quad A = \frac{e}{e-1} \tag{52}$$

$$G(t) = \int_0^1 Ae^{-x}e^{itx}\mathrm{d}x = \frac{A}{it-1}\left[e^(it-1)x\right]_0^1 = \frac{A}{it-1}\left[e^{it-1}-1\right] \tag{53}$$

$$G(0) = -A\left(e^{-1}-1\right) = -\frac{e}{e-1}e^{-1}\left(1-e\right) = 1 \tag{54}$$

$$G'(t) = -\frac{iA}{(it-1)^2}\left(e^{it-1}-1\right) + \frac{iA}{it-1}e^{it-1} \tag{55}$$

$$G'(0) = -iA\left(e^{-1}-1\right) - iAe^{-1} = -iA\left(e^{-1}-1\right) = i\frac{e-2}{e-1} \tag{56}$$

$$G''(t) = -\frac{2a}{(it-1)^3}\left(e^{it-1}-1\right) + \frac{2A}{(it-1)^2}e^{it-1} - \frac{A}{it-1}e^{it-1} \tag{57}$$

$$G''(0) = 2A\left(e^{-1}-1\right) + 2Ae^{-1} + Ae^{-1} = -\frac{2e-5}{e-1} \tag{58}$$

$$\langle x \rangle = -i \times i \frac{e-2}{e-1} = \frac{e-2}{e-1} \tag{59}$$

$$\left\langle x^2 \right\rangle = \frac{2e-5}{e-1} \tag{60}$$

$$\begin{aligned} \sigma^2 &= \frac{2e-5}{e-1} - \left(\frac{e-2}{e-1} \right)^2 \\ &= \frac{e^2 - 3e + 1}{(e-1)^2} \end{aligned} \tag{61}$$