Fractals

モデル化とシミュレーション特論
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1. Introduction
2. Fractals by repeated operations
3. Fractal dimension
4. Affine transformation
5. Simulation

Shapes and Order

- Simple shapes
- Simple periodic order
- Completely random shapes and phenomena
- Complex characteristics
  - coastlines, trees and leafs, hierarchical structure of organs, genetic information, languages, ecosystems, population, changes in stock markets, etc.
  - How to characterize these complex features.

Symmetry

- Symmetry: invariance under operation
- Uniform: invariant under translation to any directions
- Radial: invariant under rotation
- Periodic: invariant under translation with fixed length to some directions

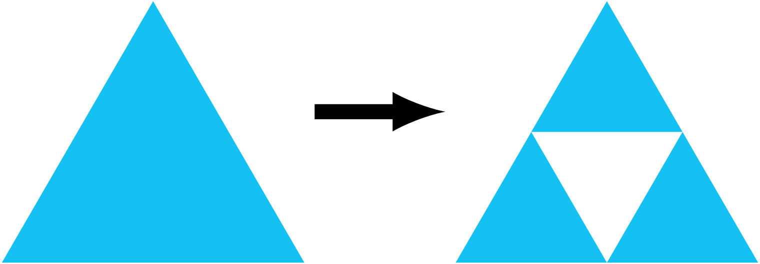
Characteristic Length

- Many natural and artificial systems have characteristic spacial or temporal length
  - crystals have lattice constants
  - periodic structure
  - color of materials: response to light with some characteristic length
- Noise: no characteristic length

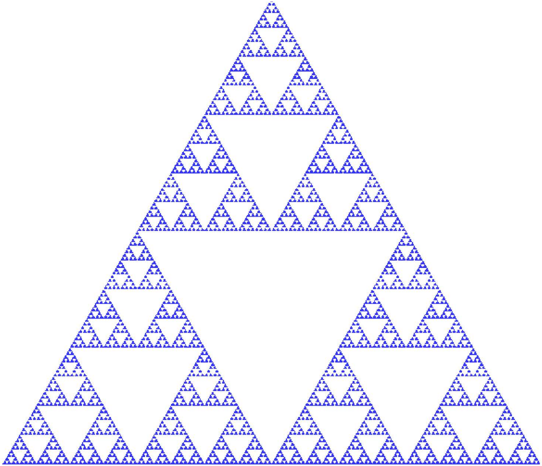
scale invariance

- Similar shapes under different scales
- invariant under expansion and reduction
- no characteristic length

Sierpinski gasket

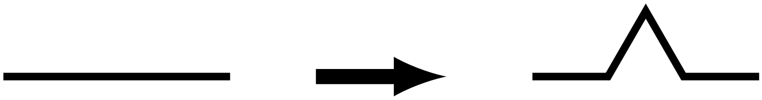


- Start from a equilateral triangle
- Hollow out the central equilateral triangle
- Hollow out the central equilateral triangles in remaining triangles.
- Repeat the operations

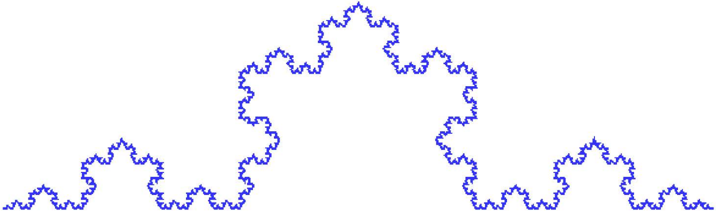


Each triangle is similar to the whole.

Koch Curve



- Start from a line
- Divide the line into three equal parts
- Put a equilateral triangle whose side is one of three parts. The triangle does not have bottom.
- Divide each line into three and put triangles at the center of each line.
- Repeat the operation



Length of Koch Curve

- Initial length $L(0) = \ell$
- Length at the first operation $L(1) = (4/3)\ell$
- Length after n operations $L(n) = (4/3)^n\ell$
- For $n \to \infty$, $L(n) \to \infty$

Area of Sierpinski Gasket

- Initial area $S(0) = s$
- Area at the first operation $S(1) = (3/4)s$
- Area after n operations $S(n) = (3/4)^n s$
- For $n \to \infty$, $S(n) \to 0$

Dimension

- We live in a 3spatial plus 1temporal dimensional space.
- Dimensions are usually integers
- Modern particle physics says that we live in a 10 or 26 dimensional space.

Topological Dimensions

- Dimension : the number of coordinates for specifying one point in a space.
- Topological Dimension
    - Point : 0 dimensional object
    - Line or curve : 1 dimensional object
    - Plane or surface: 2 dimensional object
    - Space : 3 dimensional object
    - So on

Dimensions for measurements

- Units for measurements.
- Volume : L^3
- Change unit $1/a \rightarrow$ Value of its volume changes a^3
  $1\mathrm{m}^3 = 10^6\mathrm{cm}^3$
- Dimension describes how a quantity scales with the measurement unit.

Self-similarity dimension

- A shape consists of b similar shapes which are the same as the whole shape with $1/a$ scale down.
- The fractal dimension of the shape is

$$D = \frac{\ln b}{\ln a}$$

- The shape looks similar if you look the shape with its $1/a$ scale.

Self-similarity dimensions for Koch curve and Sierpinski gasket

- Koch curve

$$D = \frac{\ln 4}{\ln 3} = 1.2618\ldots > 1$$

  thicker than a curve

- Sierpinski gasket

$$D = \frac{\ln 3}{\ln 2} = 1.58496\ldots < 2$$

  thinner than a plane

Hausdorff Measure

- A shape S is covered with enumerable shapes $u_0, u_1, \ldots$.
- Those diameters $U_0, U_1, \ldots$ are less than $L > 0$.
- The hausdorff measure of the shape S is defined as

$$H^d(S) = \lim_{L \to 0} \inf_{U_i < L} \left(\sum_i |U_i|^d \right)$$

Hausdorff dimension

- By decreasing the value of d from infinite, there is the critical value where the Hausdorff measure jumps from zero to infinite.
- It is called the Hausdorff dimension.

Capacity dimension

- Self-similarity dimension
  - Applicable only for shapes with complete self-similarity.
- Hausdorff dimension
  - Includes limit operations.
  - Difficult for applying for realistic cases
- Need effective methods applicable for observations and simulations.
- Fractal dimension with statistical meanings.

Capacity dimension

- A shape is covered with b similar shapes with $1/a$ scale-down.
- The capacity dimension D_c is defined as

$$D_c = \frac{\ln b}{\ln a}$$

Box-Counting method

- Fractal dimension for data
- 2 dimensional cases
  - Squares covering the shape : linear size ℓ
  - The number of squares : $n(\ell)$
  - Change its size to ℓ/m
  - repeat
- Plot $n(\ell)$ against ℓ in log-log plot.
- Fractal dimension : slope of the line

Affine transformation

- rotation, scaling, shear (剪断), translation

$$\vec{x} \mapsto A\vec{x} + \vec{b} \tag{1}$$

- Express as a map $W : X \to X$
- Consider a set of maps: $\{W_i\}$
- Fixed point of the map: for a set of points $U \subset X$

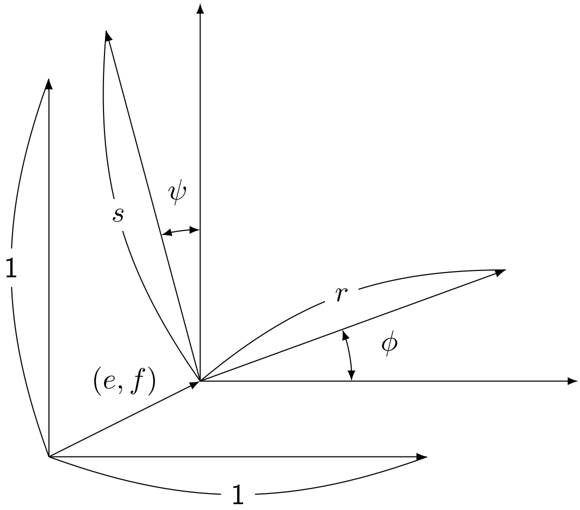
$$\bigcup_i W_i\left(U\right) = U \tag{2}$$

Expressions of Affine transformation

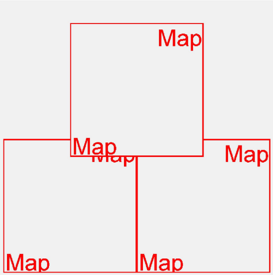
- $L \times L$ initial image
- parameter set : (r, s, ϕ, ψ, e, f)

$$\vec{x} \mapsto \begin{pmatrix} r\cos\phi & -s\sin\psi \\ r\sin\phi & s\cos\psi \end{pmatrix} \vec{x} + \begin{pmatrix} eL \\ fL \end{pmatrix}$$

Affine Parameters

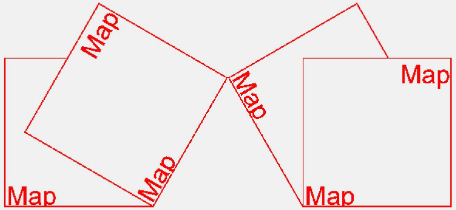


Sierpinski gasket



$$
\{(r, s, \phi, \psi, e, f)\} = \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{\sqrt{3}}{4}\right) \right\}
$$

Koch curve



$$
\begin{aligned}
&\{(r, s, \phi, \psi, e, f)\} \\
&= \left\{ \left(\frac{1}{3}, \frac{1}{3}, 0, 0, 0, 0 \right), \left(\frac{1}{3}, \frac{1}{3}, \frac{\pi}{3}, \frac{\pi}{3}, 0, 0 \right), \right. \\
&\quad \left. \left(\frac{1}{3}, \frac{1}{3}, -\frac{\pi}{3}, -\frac{\pi}{3}, \frac{1}{2}, \frac{1}{3} \sin\left(\frac{\pi}{3}\right) \right), \left(\frac{1}{3}, \frac{1}{3}, 0, 0, \frac{2}{3}, 0 \right) \right\}
\end{aligned}
$$

Affine transformation in Java

- Built-in `AffineTransform` class
  - initialize with affine parameters (r, s, ϕ, ψ, e, f)
- Preparing operation
  - `AffineTransformOp` class
  - Needs a `AffineTransform` instance for initialization
- Transforming images
  - `AffineTransformOp.filer()` method

Classes

- `AbstractFractal` class
  - Initialize image
  - Update: Affine transformation
  - Show map
- Each fractal class only defines Affine transformation.

Sample program

`https://github.com/modeling-and-simulation-mc-saga/AffineFractals`