Chaos and Logistic Map

モデル化とシミュレーション特論
2021 年度前期
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Chaos

- Henri Poincaré
  - complex trajectories for 3-body problems (1880's)
- Edward Lorenz
  - difficulties in weather forecasts (1960's)
  - small initial differences expands.
- Turbulence
- Logistic Map as a simplest chaos model
  routes to chaos, intermittency, band splitting, etc.

Logistic Map

- A species which has off-springs
- If the number of individuals small, the number of off-springs will increase proportionally.
- If large, the number of off-springs will decrease because of restriction from the environment.

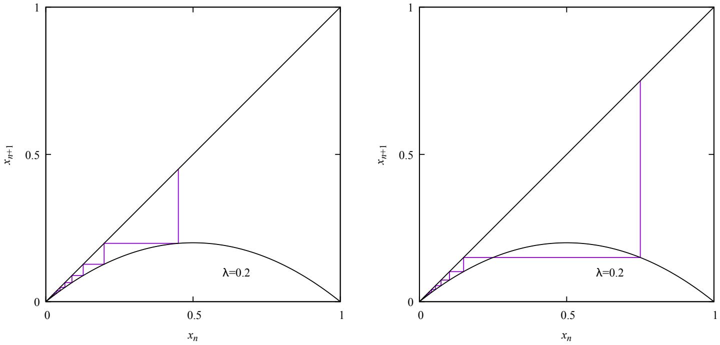
$$x_{n+1} = f_\lambda(x_n) \tag{1}$$

$$f_\lambda(x) = 4\lambda x(1-x) \tag{2}$$

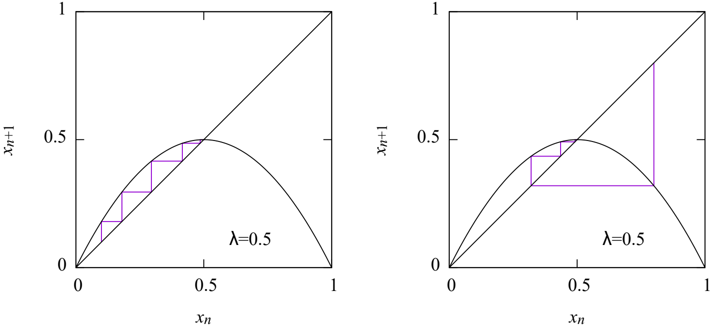
$$x_i \in [0,1], \qquad \lambda \in [0,1]$$

fixed points for small λ

- fixed points are solutions of $x = f_\lambda(x)$
- $\lambda < 1/4$
  - only one fixed point at $x = 0$
  - example $\lambda = 0.2$



- $1/4 < \lambda < 3/4$
  - two fixed points at $x = 0$ and $(4\lambda - 1)\,/\,(4\lambda)$
  - trajectories do not go to $x = 0$
  - example $\lambda = 0.5$ from $x_0 = 0.1$ and 0.8



stability of fixed points

- A point $x_0 = x_\mathrm{f} + \delta$ near a fixed point x_f

$$x_1 = f_\lambda\left(x_\mathrm{f} + \delta\right) = f_\lambda\left(x_\mathrm{f}\right) + \delta \left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_\mathrm{f}} + O\left(\delta^2\right) \tag{3}$$

- stable: $\left|\mathrm{d}f_\lambda/\mathrm{d}x\right| < 1$
- unstable: $\left|\mathrm{d}f_\lambda/\mathrm{d}x\right| > 1$

Stability of $x_\mathrm{f} = 0$

$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=0} = 4\lambda\left.(1-2x)\right|_{x=0} = 4\lambda \tag{4}$$

- stable: $\lambda < 1/4$
- unstable: $\lambda > 1/4$

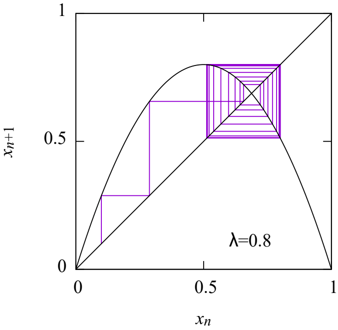
Stability of $x_{\mathrm{f}} = (4\lambda - 1) / (4\lambda)$

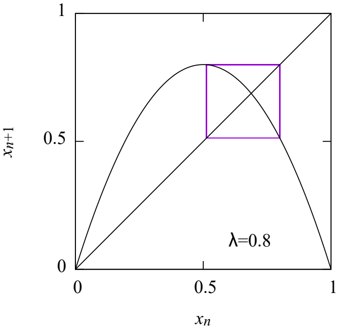
$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_{\mathrm{f}}} = 4\lambda\,(1-2x)|_{x=x_{\mathrm{f}}} = 2 - 4\lambda \tag{5}$$

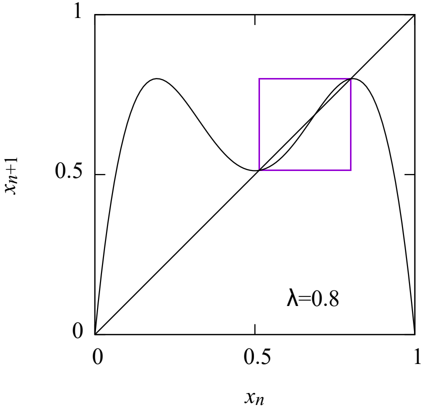
- $|\mathrm{d}f_\lambda/\mathrm{d}x| = 1$ at $\lambda = 1/4$
- $|\mathrm{d}f_\lambda/\mathrm{d}x| = -1$ at $\lambda = 3/4$
- stable: $1/4 < \lambda < 3/4$

Period Doubling

- at $\lambda = 3/4$ period-2 trajectory appears







$$x_{\pm} = f_{\lambda}\left(x_{\mp}\right) = \frac{1}{8\lambda}\left[4\lambda + 1 \pm \sqrt{\left(4\lambda+1\right)\left(4\lambda-3\right)}\right] \quad (6)$$

Stabitily of period-2 trajectories

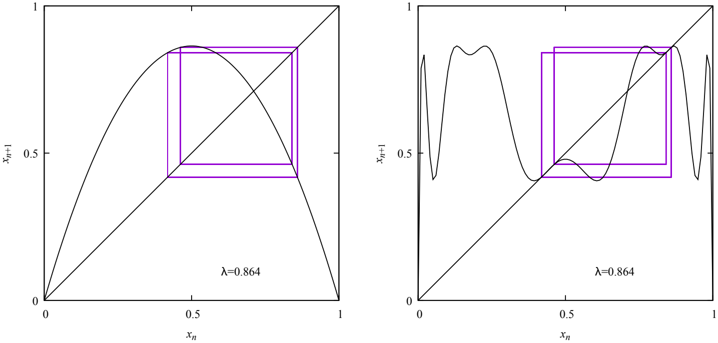
$$f_\lambda^{[n+1]}(x) = f_\lambda\left(f_\lambda^{[n]}(x)\right) \tag{7}$$

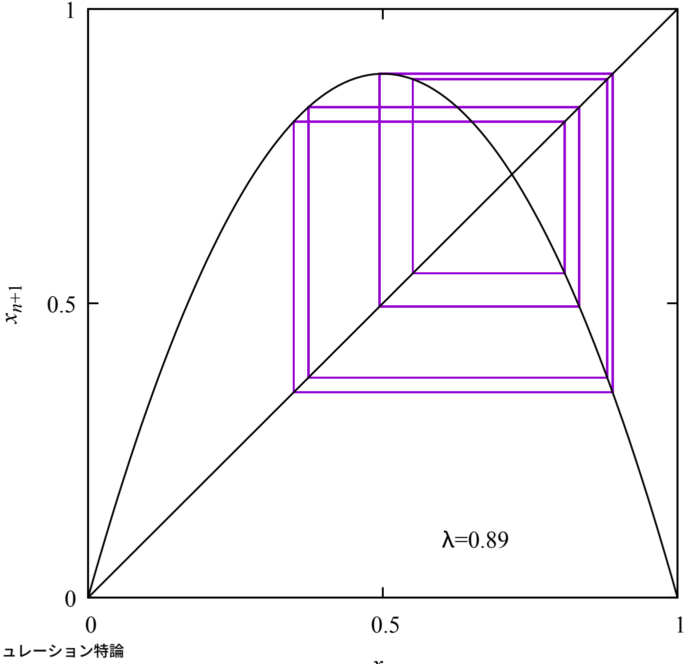
$$f_\lambda^{[1]}(x) = f_\lambda(x) \tag{8}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[2]}\right|_{x=x_\pm} = 1 - (4\lambda + 1)(4\lambda - 1) \tag{9}$$

- the next instability

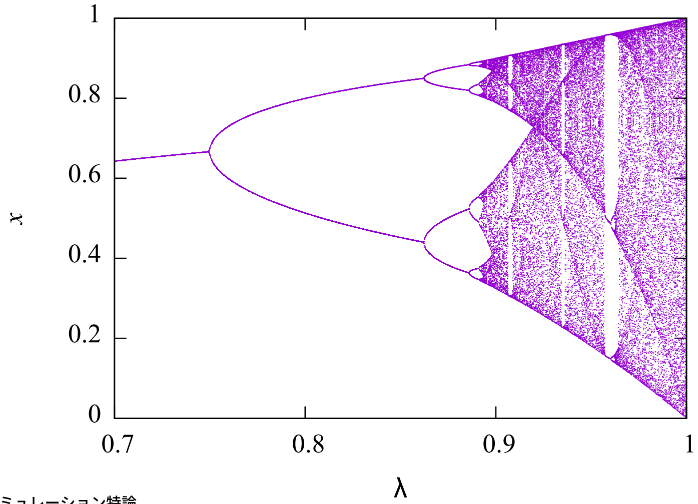
$$\lambda = \frac{1+\sqrt{6}}{4} \simeq 0.8624 \tag{10}$$


1
0.5
0
0
0.5
1
x_{n+1}
x_n
λ=0.864
1
0.5
0
0
0.5
1
x_{n+1}
x_n
λ=0.864


1
0.5
0
x_{n+1}
λ=0.89
0
0.5
1

Period Doubling to Chaos

- Trajectories are doubled by increasing λ
- Period becomes infinite at $\lambda \simeq 0.893$



Sample Programs

`https://github.com/modeling-and-simulation-mc-saga/Logistic`

- `model/Logistic.java`
  - Logistic map
  - setting λ
  - `update()` method
- `analysis/PrintOrbit.java`
  - show orbits in (x_n, x_{n+1})-plane
  - show Logistic map : $f_\lambda^{[n]}(x)$
  - Output orbits in pdf through gnuplot

Direct output to PDF

- `utils/Gnuplot.java`
    - open `gnuplot` as a process
    - open `outputstream` of the process
    - write `gnuplot` commands to the stream
    - You have to set the path to `gnuplot`.

Gnuplot with standard input

- Input from standard input

```
plot "-"
```

- Script containing data

```
plot "-"
1 2
3 5
6 10
10 7
end
```