Fractals

モデル化とシミュレーション特論
2023 年度前期
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Shapes and Order

- Simple shapes
- Simple periodic orders
- Completely random shapes and phenomena
- Complex characteristics
  - coastlines, trees and leafs, hierarchical structure of organs, genetic information, languages, ecosystems, changes in stock markets, etc.
  - How can we characterize these complex features.
- Let us see some images by searching with a keyword *fractal*.

Sample program
`https://github.com/modeling-and-simulation-mc-saga/AffineFractals`

Symmetry: 対称性

- Symmetry: invariance under operations
- Uniform: invariant under translation in any directions
- Radial: invariant under rotation
- Periodic: invariant under translation with a fixed length to fixed directions
- tiling without repeats
  `https://www.newscientist.com/article/2365363-mathematicians-discover-shape-that-can-tile-a`

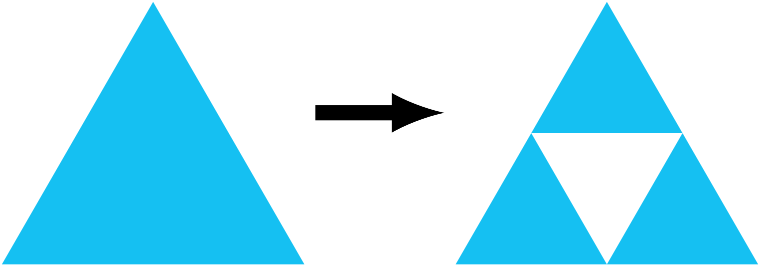
Characteristic Length: 特徴的長さ

- Many natural and artificial systems have characteristic spacial or temporal lengths
  - Crystals have lattice constants, representing their periodic structure
  - The color of materials correspond to light of some characteristic wavelengths
- Noise does not have characteristic lengths
- Solar light
  `https://www.e-education.psu.edu/meteo300/node/683`

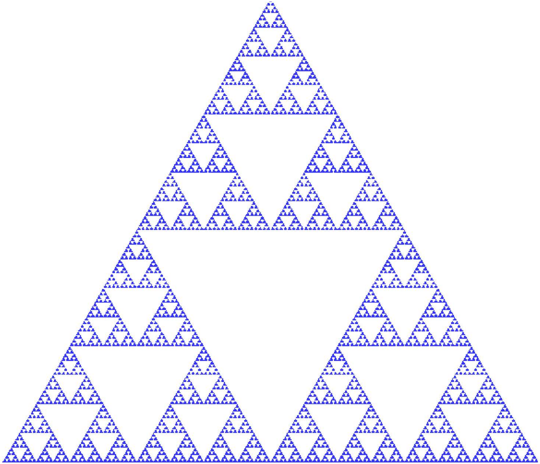
Scale Invariance

- Invariant under expansion and reduction
- Similar shapes across different scales
- No characteristic lengths
- Some special distribution of scales

Sierpinski gasket

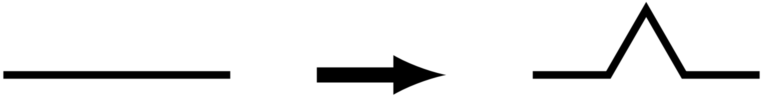


- Start from a equilateral triangle
- Remove the central equilateral triangle
- Remove the central equilateral triangles in remaining triangles.
- Repeat the operations

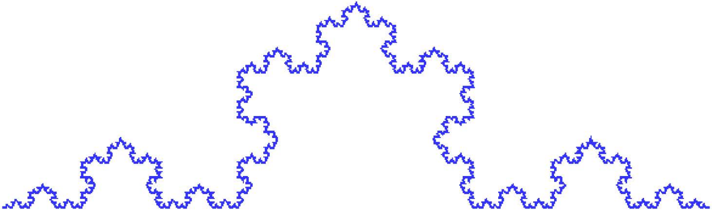


Each triangle is similar to the entire structure.

Koch Curve



- Start from a straight line.
- Divide the line into three equal lengthsegments.
- Place a equilateral triangle with a side length equal to one of three parts at the central segment.
  - The triangle does not have bottom side.
- Divide each line into three segments and place triangles at the each central segment.
- Repeat the operation



Length of Koch Curve

- Assume the initial length as $L(0) = \ell$
- Length at the first operation $L(1) = (4/3)\ell$
- Length after n operations $L(n) = (4/3)^n\ell$
- For $n \to \infty$, $L(n) \to \infty$
  - An curve with *infinite length* lives in a finite area!

Area of Sierpinski Gasket

- Assume the initial area as $S(0) = s$
- Area at the first operation $S(1) = (3/4)s$
- Area after n operations $S(n) = (3/4)^n s$
- For $n \to \infty$, $S(n) \to 0$
    - The area goes to *zero*!

Dimension: 次元

- We believe living in a 3 spatial plus 1 temporal dimensional space.
- Dimensions are specified usually by integers
- Modern particle physics says that we live in a 10 or 26 dimensional space.

Topological Dimensions

- Dimension: the number of coordinates for specifying one point in a space.
- Topological Dimension
  - Point: 0 dimensional object
  - Line or curve: 1 dimensional object
  - Plane or surface: 2 dimensional object
  - Space: 3 dimensional object
  - So on

Dimensions considered by measurements

- Units for measurements.
- Volume : L^3
- Change unit $1/a \rightarrow$ Value of its volume changes a^3
  ex. $1\mathrm{m}^3 = 10^6\mathrm{cm}^3$
- Magnify the linear scale by a: The volume becomes a^3 times larger.
- Dimension describes how a quantity scales with the measurement unit.

Self-similarity dimension

- A shape consists of b similar shapes, each of which is identical to the whole shape but scaled down by a factor $1/a$.
- The fractal dimension of the shape is

$$D = \frac{\ln b}{\ln a} \tag{4.1}$$

- The shape appears similar when viewed at a scale $1/a$.
- Square $b = a^2$

$$D = \frac{2 \ln a}{\ln a} = 2$$

Self-similarity dimensions for Koch curve and Sierpinski gasket

- Koch curve

$$D = \frac{\ln 4}{\ln 3} = 1.2618\ldots > 1$$

  thicker than a curve

- Sierpinski gasket

$$D = \frac{\ln 3}{\ln 2} = 1.58496\ldots < 2$$

  thinner than a plane

Hausdorff Measure

- A shape S is covered with enumerable shapes $u_0, u_1, \ldots$.
- Those diameters $U_0, U_1, \ldots$ are less than $L > 0$.
- The hausdorff measure of the shape S is defined as

$$H^d(S) = \lim_{L \to 0} \inf_{U_i < L} \left(\sum_i |U_i|^d \right)$$

Hausdorff dimension

- As the value of d decreases from infinity, there exists a critical value at which the Hausdorff measure transitions from zero to infinity.
- The critical value is called the Hausdorff dimension.

Capacity dimension

- Self-similarity dimension
  - Applicable only for shapes with complete self-similarity.
- Hausdorff dimension
  - Includes limit operations.
  - Difficult for applying for realistic cases
- Need effective methods applicable for both observations and simulations.
- Fractal dimension provides a statistical interpretation.

Capacity dimension

- A shape is covered with b similar shapes scaled down by a factor $1/a$.
- The capacity dimension D_c is defined as

$$D_c = \frac{\ln b}{\ln a}$$

Box-Counting method

- Fractal dimension for data
- 2 dimensional cases
  - Squares covering the shape : linear size ℓ
  - The number of squares : $n(\ell)$
  - Change its size to ℓ/m
  - repeat
- Plot $n(\ell)$ against ℓ in log-log plot.
- Fractal dimension : slope of the line

Affine transformation

- rotation, scaling, shear (剪断), translation

$$\vec{x} \mapsto A\vec{x} + \vec{b} \tag{5.1}$$

- Express as a map $W : X \to X$
- Consider a set of maps: $\{W_i\}$
- Fixed point of the map: for a set of points $U \subset X$

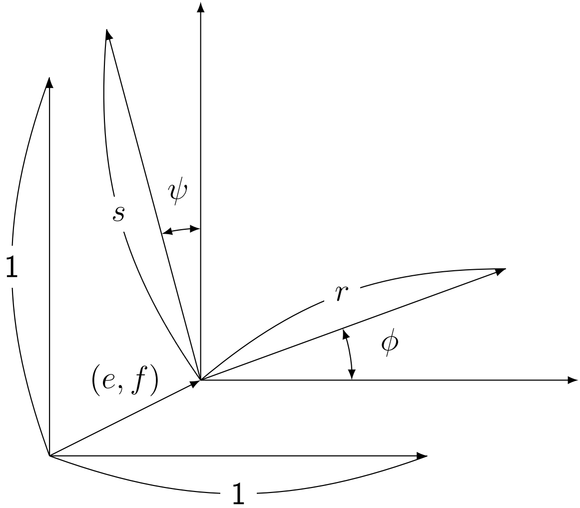
$$\bigcup_i W_i\,(U) = U \tag{5.2}$$

Expressions of Affine transformation

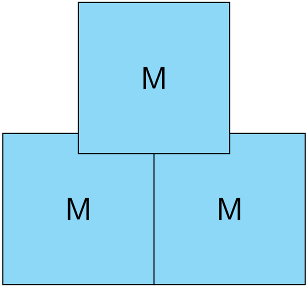
- $L \times L$ initial image
- parameter set : (r, s, ϕ, ψ, e, f)

$$\vec{x} \mapsto \begin{pmatrix} r\cos\phi & -s\sin\psi \\ r\sin\phi & s\cos\psi \end{pmatrix} \vec{x} + \begin{pmatrix} eL \\ fL \end{pmatrix}$$

Affine Parameters

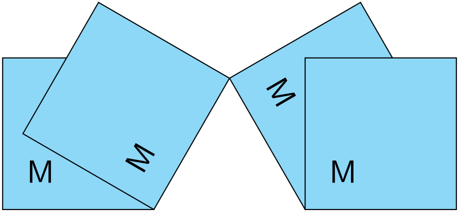


Sierpinski gasket



$$\{(r, s, \phi, \psi, e, f)\}$$
$$= \left\{ \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2}, 0, 0, \frac{1}{4}, \frac{\sqrt{3}}{4}\right) \right\}$$

Koch curve



$$\{(r,s,\phi,\psi,e,f)\}$$
$$=\left\{\left(\frac{1}{3},\frac{1}{3},0,0,0,0\right),\left(\frac{1}{3},\frac{1}{3},\frac{\pi}{3},\frac{\pi}{3},0,0\right),\right.$$
$$\left.\left(\frac{1}{3},\frac{1}{3},-\frac{\pi}{3},-\frac{\pi}{3},\frac{1}{2},\frac{1}{3}\sin\left(\frac{\pi}{3}\right)\right),\left(\frac{1}{3},\frac{1}{3},0,0,\frac{2}{3},0\right)\right\}$$

Affine transformation in Java

- Built-in `AffineTransform` class
  - initialize with affine parameters (r, s, ϕ, ψ, e, f)
- Preparing operation
  - `AffineTransformOp` class
  - Needs a `AffineTransform` instance for initialization
- Transforming images
  - `AffineTransformOp.filer()` method

Classes

- `AbstractFractal` class
  - Initialize image
  - Update: Affine transformation
  - Show map
- Each fractal class only defines Affine transformation.

Mandelbrot Set

- Consider a complex numbers c and a series

$$z_0 = c \tag{7.1}$$

$$z_{n+1} = z_n^2 + c \tag{7.2}$$

- The Mandelbrot set M is defined as a set of complex numbers c for which the sequence z_∞ remains bounded.

`https://github.com/modeling-and-simulation-mc-saga/Mandelbrot`