Chaos and Logistic Map

モデル化とシミュレーション特論
2023 年度前期

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Introcution: Where chaoses live?

- Henri Poincaré
  - Complex trajectories for 3-body problems (1880's)
- Edward Lorenz
  - Difficulties in weather forecasts (1960's)
  - Small initial differences expands.

  https://www.google.com/search?q=Lorenz+model
- Turbulence: Kármán's vertex
  https://www.google.com/search?q=karman+vortex
- Logistic Map as a simplest chaos model
  routes to chaos, intermittency, band splitting, etc.

Logistic Map

- A simple model of population dynamics.
- A species which has off-springs
- If the number of individuals small, the number of off-springs will increase proportionally.
- If it is large, the number of off-springs will decrease due to the environmental constraints.

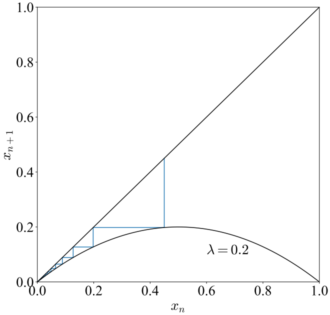
$$x_{n+1} = f_\lambda(x_n) \tag{2.1}$$

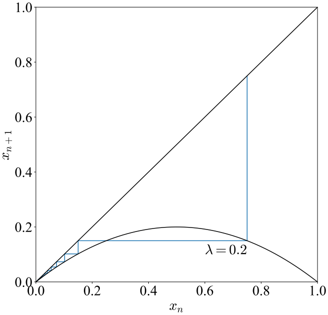
$$f_\lambda(x) = 4\lambda x(1-x) \tag{2.2}$$

$$x_i \in [0,1], \qquad \lambda \in [0,1]$$

Fixed points for small $\lambda < 1/4$

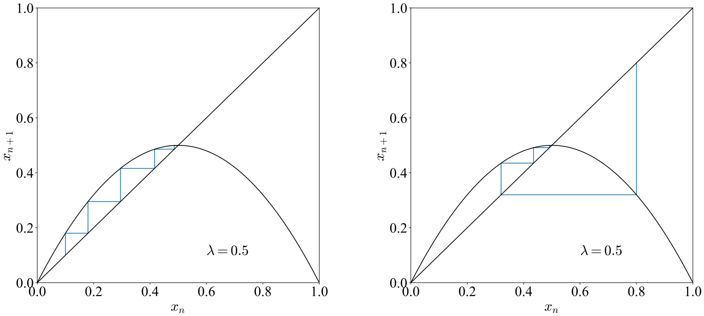
- Fixed points are solutions of $x = f_\lambda(x)$
  - The size of the population remains unchanged.
- $\lambda < 1/4$
  - Only one fixed point at $x = 0$
  - The population extincts.
  - Example $\lambda = 0.2$





$1/4 < \lambda < 3/4$

- two fixed points at $x = 0$ and $(4\lambda - 1) / (4\lambda)$
- trajectories do not go to $x = 0$
- example $\lambda = 0.5$ from $x_0 = 0.1$ and 0.8



Stability of fixed points

- A point $x_0 = x_\mathrm{f} + \delta$ near a fixed point x_f

$$x_1 = f_\lambda\left(x_\mathrm{f} + \delta\right) = f_\lambda\left(x_\mathrm{f}\right) + \delta \left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_\mathrm{f}} + O\left(\delta^2\right) \tag{2.3}$$

- Stable: $|\mathrm{d}f_\lambda/\mathrm{d}x| < 1$
  - Deviation from the fixed point decreases
- Unstable: $|\mathrm{d}f_\lambda/\mathrm{d}x| > 1$
  - Deviation from the fixed point increases

Stability of $x_\mathrm{f} = 0$

$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=0} = 4\lambda\left.(1-2x)\right|_{x=0} = 4\lambda \tag{2.4}$$

- Stable: $\lambda < 1/4$
- Unstable: $\lambda > 1/4$

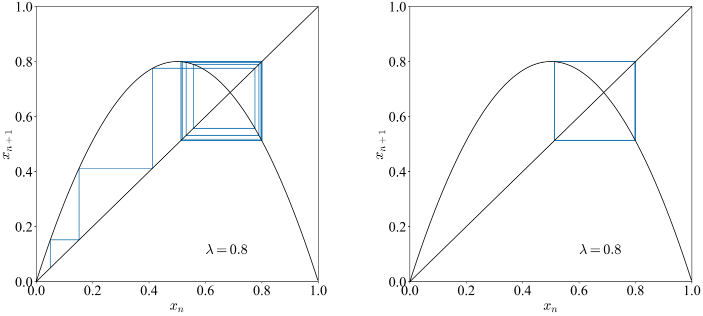
Stability of $x_\mathrm{f} = (4\lambda - 1) \,/\, (4\lambda)$

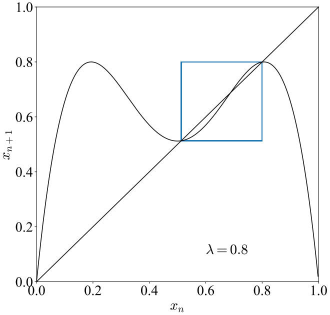
$$\left.\frac{\mathrm{d}f_\lambda}{\mathrm{d}x}\right|_{x=x_\mathrm{f}} = 4\lambda\left(1 - 2x\right)\big|_{x=x_\mathrm{f}} = 2 - 4\lambda \tag{2.5}$$

- $\mathrm{d}f_\lambda/\mathrm{d}x = 1$ at $\lambda = 1/4$
- $\mathrm{d}f_\lambda/\mathrm{d}x = -1$ at $\lambda = 3/4$
- Stable: $1/4 < \lambda < 3/4$

Period Doubling

- Period-2 trajectory appears at $\lambda = 3/4$





- The curve shows $f_\lambda\left(f_\lambda\left(x\right)\right)$.
- Two crossing points between $y = x$ and $y = f_\lambda(f_\lambda(x))$

$$\begin{aligned} x_\pm &= f_\lambda\left(f_\lambda\left(x_\pm\right)\right) = f_\lambda\left(x_\mp\right) \\ &= \frac{1}{8\lambda}\left[4\lambda + 1 \pm \sqrt{\left(4\lambda+1\right)\left(4\lambda-3\right)}\right] \end{aligned} \tag{3.1}$$

Stabitily of period-2 trajectories

$$f_\lambda^{[n+1]}(x) = f_\lambda\left(f_\lambda^{[n]}(x)\right) \tag{3.2}$$

$$f_\lambda^{[1]}(x) = f_\lambda(x) \tag{3.3}$$

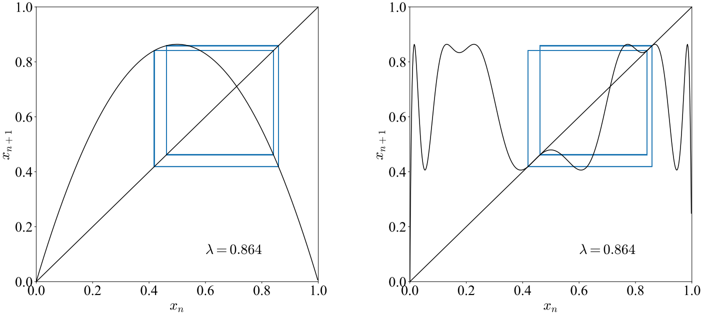
$$\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[2]}(x) = f'_\lambda(f_\lambda(x))\, f'_\lambda(x) \tag{3.4}$$

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[2]}\right|_{x=x_\pm} &= 4\lambda(1-2x_\pm)\,4\lambda(1-2x_\mp) \\ &= 1-(4\lambda+1)(4\lambda-3)\end{aligned} \tag{3.5}$$

- The next instability

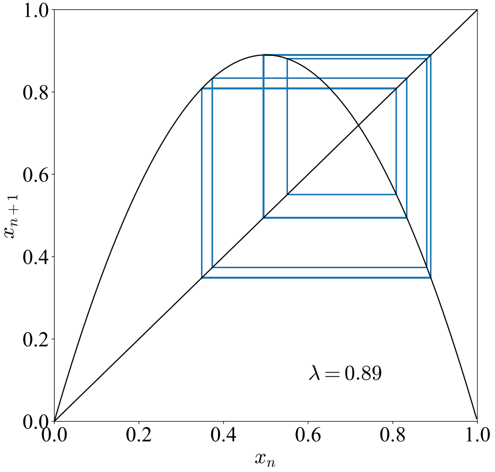
$$\lambda = \frac{1+\sqrt{6}}{4} \simeq 0.8624 \tag{3.6}$$

Period 4 trajectory



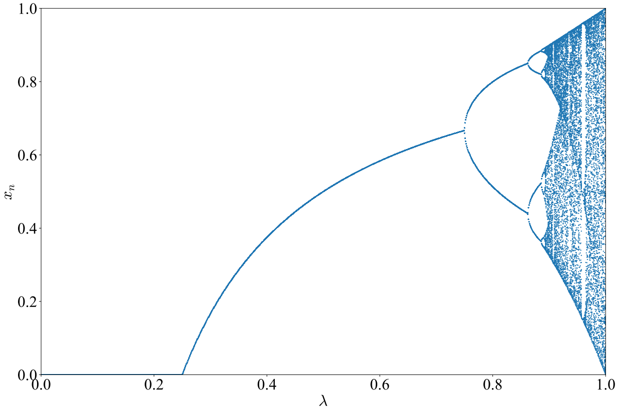
The curve in the right figure is $f_\lambda^{[4]}(x)$

Period 8 trajectory



Period Doubling to Chaos

- Trajectories are doubled by increasing λ
- Period becomes infinite at $\lambda \simeq 0.893$



Sample Programs

`https://github.com/modeling-and-simulation-mc-saga/Logistic`

- `model/Logistic.java`
  - Logistic map
  - setting λ
  - `update()` method
- `analysis/PrintTrajectory.java`
  - show trajectory in (x_n, x_{n+1})-plane
  - show Logistic map : $f_{\lambda}^{[n]}(x)$