Chaos and Logistic Map : part2

モデル化とシミュレーション特論
2023 年度前期

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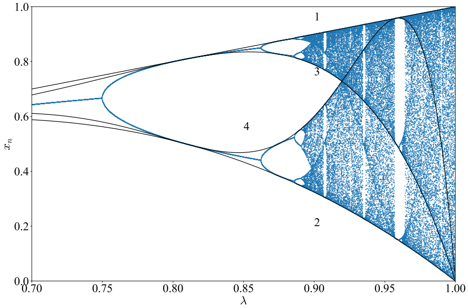
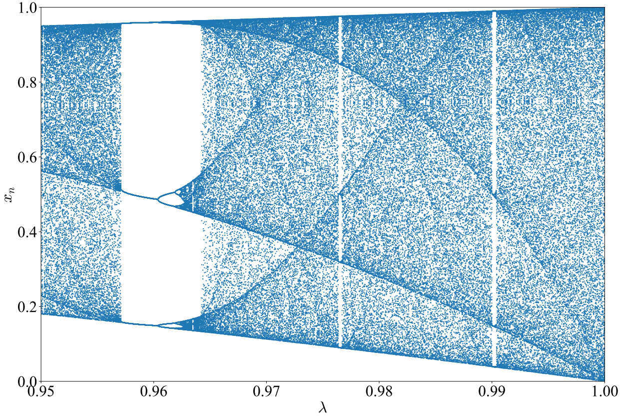
Period doubling to chaos

- Trajectories are repeatedly doubled by increasing λ
  - $2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \cdots$
- Period becomes infinite at $\lambda \simeq 0.893$
  - No longer periodic



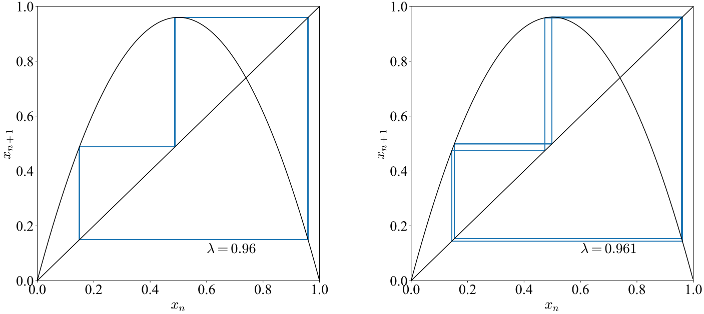
- For $\lambda > 0.893$, trajectories exhibit a band structure.
  - Not periodic
  - Not random: Non-uniform density of trajectories
- Certain windows are identifiable
  - $k \times 2^n$ period (k: primary numbers)

Period-3 region



You can also see period-5 and 7 windows.

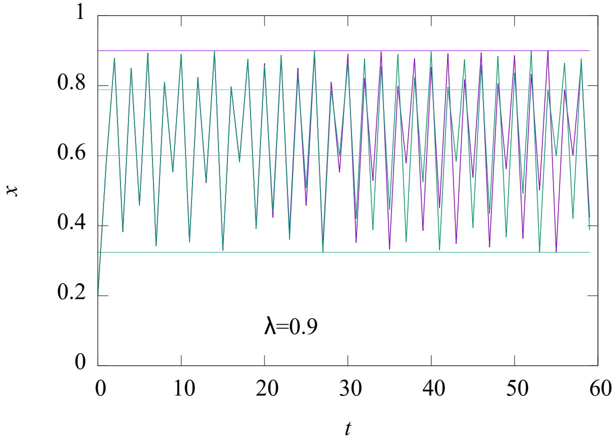
Period-3 orbit



- Period-3 trajectories occur near $\lambda \sim 0.96$
- Those trajectories double in period to become period-6 trajectories

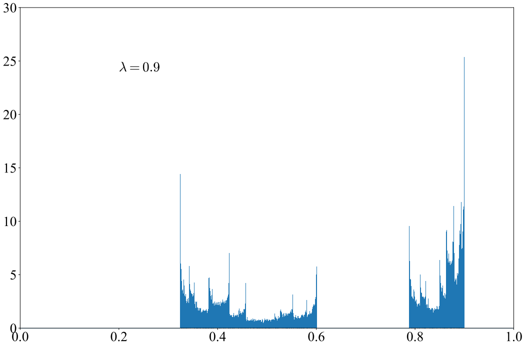
Chaotic motions

- A small difference in initial values expands.
- Eventually resulting in two trajectories that appears to behave independently.



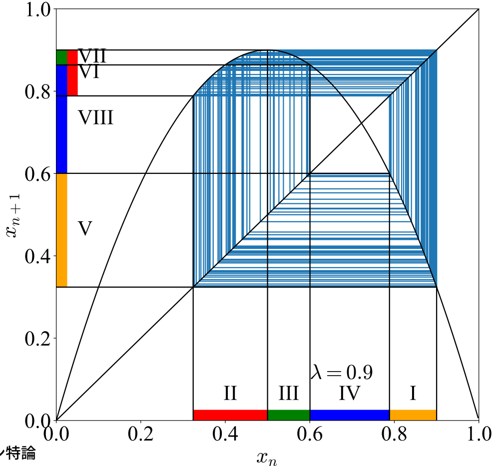
Non-uniform density of trajectories

$$f_\lambda^{[n]}\left(\frac{1}{2}\right)$$

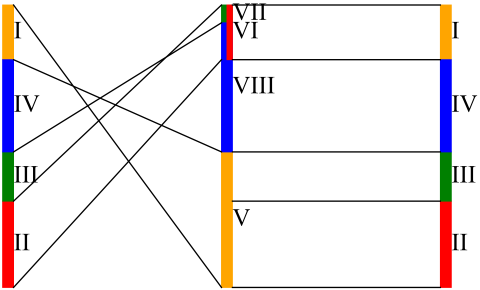


Bands of trajectories

- Bands of trajectories are expended and folded.
- This is the origin of chaotic motion.

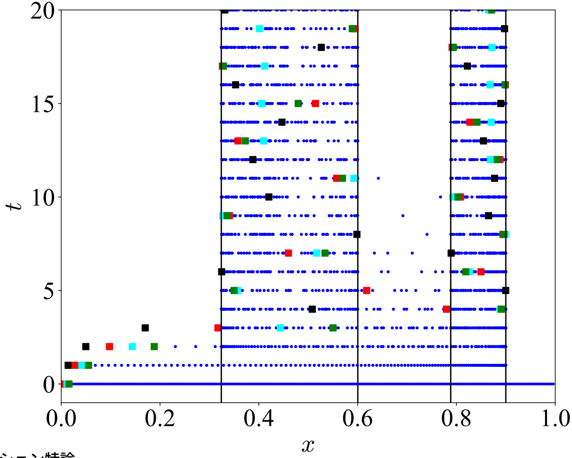


Folding and overlapping bands



Uniform initial points are absorbed into two bands

- Two points ■ and ■, which are initially close each other, separate and behave almost independently.



Super-stable point: $x = 1/2$

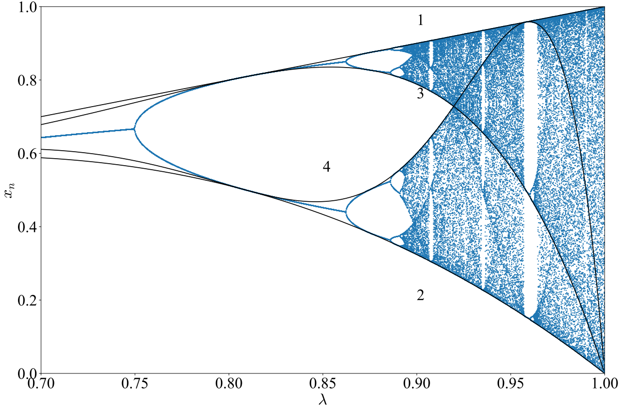
$$
\begin{aligned}
f_\lambda(x) &= 4\lambda(1-x) \\
f'_\lambda(x) &= 4\lambda(1-2x) \\
\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[2]}(x) &= f'_\lambda(f_\lambda(x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f_\lambda(x) \qquad (3.1) \\
\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[n]}(x) &= f'_\lambda\left(f_\lambda^{[n-1]}(x)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[n-1]}(x) \qquad (3.2)
\end{aligned}
$$

$$f'_\lambda\left(\frac{1}{2}\right) = 4\lambda\left(1 - 2\frac{1}{2}\right) = 0 \tag{3.3}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[2]}(x)\bigg|_{x=x_0=1/2} &= f'_\lambda\left(f_\lambda(x)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f_\lambda(x)\bigg|_{x=x_0=1/2} \\ &= f'_\lambda(x_1) \cdot f'_\lambda(x_0) = 0\end{aligned} \tag{3.4}$$

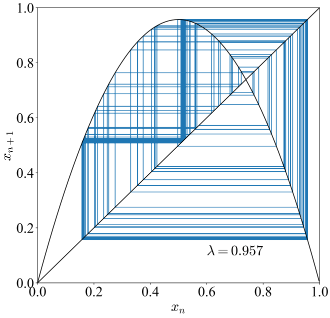
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[n]}(x)\bigg|_{x=x_0=1/2} &= f'_\lambda\left(f_\lambda(x_{n-1})\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x} f_\lambda^{[n-1]}(x)\bigg|_{x=x_0=1/2} \\ &= \prod_{k=0}^{n-1} f'_\lambda(x_k) = 0\end{aligned} \tag{3.5}$$

Trajectories of $x = 1/2$ are keys to understand band structure

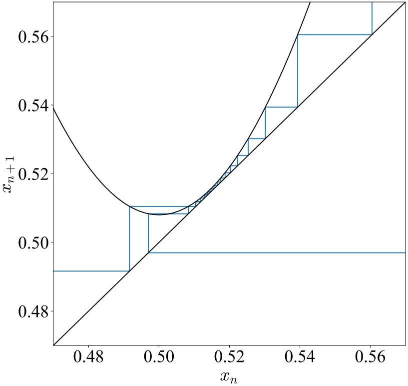


Tangent Bifurcation

- λ_{C} : period-3 trajectories emerges
- A little bit lower λ than λ_{C}
- $f_{\lambda}^{[3]}(x)$ does not intersect with $y = x$ line. There are narrow corridor.

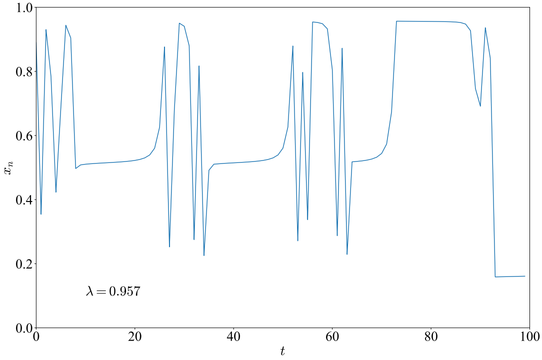


- Trajectories (per 3 times) stays long time at the narrow corridor



Intermittency (間欠)

- After staying in the narrow corridor, trajectories varies widely.



Note: x values are plotted every 3 steps.