非決定性有限オートマトンと決定性有限オートマトン

離散数学・オートマトン
2021 年後期
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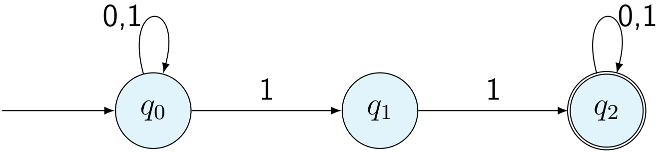
非決定性有限オートマトン: 復習

$$M = \langle Q, \Sigma, \delta, q_0, F \rangle \tag{1}$$

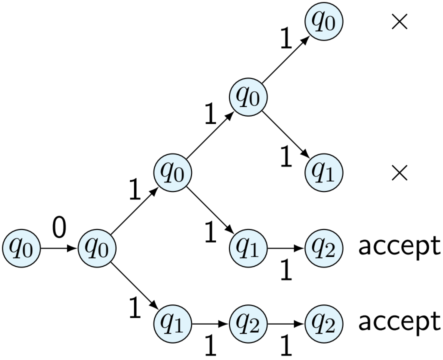
- Q : 内部状態の集合
- Σ : 入力アルファベット
- $\delta : Q \times \Sigma \to 2^Q$: 状態遷移関数
- $q_0 \in Q$: 初期状態
- $F \subseteq Q$: 受理状態

例 1



入力 w を受理するとは、w によって引き起こされた状態遷移の遷移先のなかに、受理状態 F の要素が含まれていること。

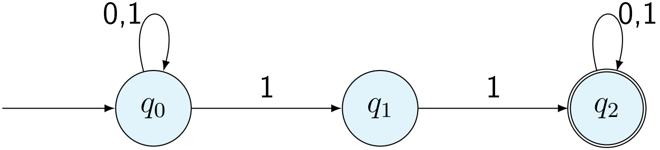
入力 0111



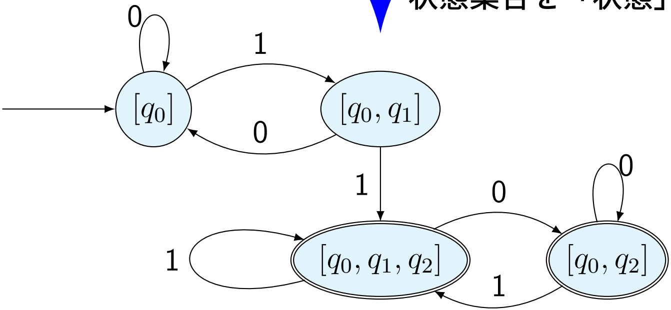
NFA から DFA へ

- NFA に対する DFA が構成できる
- つまり、NFA と DFA はその能力に差が無い
- 「非決定性」の拡張として、入力無しでの動作 (ϵ-動作) を導入
- DFA に対応した NFA を作ることができることは自明

例 1：NFA から DFA へのイメージ



状態集合を「状態」とみなす



NFA$M = \langle Q, \Sigma, \delta, q_0, F \rangle$ に対応した DFA M'

$$M' = \langle Q', \Sigma, \delta', [q_0], F' \rangle \tag{2}$$

- $Q' \in 2^Q$: 集合と区別するために [] を使う
- Σ : 入力アルファベット
- $\delta' : Q' \times \Sigma \to Q'$: 状態遷移関数
- $[q_0] \in Q'$: 初期状態
- $F' = \left\{ A \in 2^Q \mid A \cap F \neq \emptyset \right\}$: 受理状態

Algorithm 1 Q' と δ' を構成するアルゴリズム

```
δ′ = ∅, Q′ = ∅, Q′_work.push([q_0])            ▷ Q′_work は待ち行列
while |Q′_work| > 0 do
    q′ = Q_work.pop()
    for all a ∈ Σ do
        q′_new = ∅
        for all q ∈ q′ do
            for all p ∈ δ(q, a) do
                q′_new.append({p})
            end for
        end for
        δ′(q′, a) = q′_new
        if (q′_new ∉ Q′) ∧ (q′_new ∉ Q′_work) then
            (Q′_work.push(q′_new)
        end if
    end for
    Q′.append(q′)
end while
```

例 1: Q' と δ' の構成

- $[q_0]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_0, 1) &= \{q_0, q_1\} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0], 0) &= [q_0] \\ \delta'([q_0], 1) &= [q_0, q_1] \end{aligned}$$

- $[q_0, q_1]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_1, 0) &= \emptyset \\ \delta(q_0, 1) &= \{q_0, q_1\} \\ \delta(q_1, 1) &= \{q_2\} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0, q_1], 0) &= [q_0] \\ \delta'([q_0, q_1], 1) &= [q_0, q_1, q_2] \end{aligned}$$

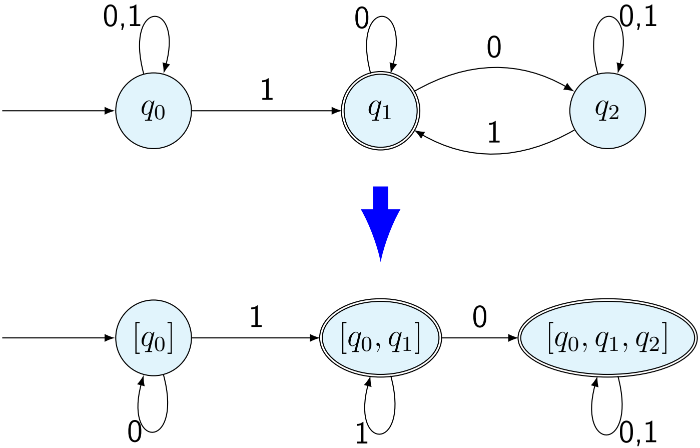
- $[q_0, q_1, q_2]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_1, 0) &= \emptyset \\ \delta(q_2, 0) &= \{q_2\} \\ \delta(q_0, 1) &= \{q_0, q_1\} \\ \delta(q_1, 1) &= \{q_2\} \\ \delta(q_2, 1) &= \{q_2\} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0, q_1, q_2], 0) &= [q_0, q_2] \\ \delta'([q_0, q_1, q_2], 1) &= [q_0, q_1, q_2] \end{aligned}$$

- $[q_0, q_2]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_2, 0) &= \{q_2\} \\ \delta(q_0, 1) &= \{q_0, q_1\} \\ \delta(q_2, 1) &= \{q_2\} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0, q_2], 0) &= [q_0, q_2] \\ \delta'([q_0, q_2], 1) &= [q_0, q_1, q_2] \end{aligned}$$

例 2



- $[q_0]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_0, 1) &= \{q_0, q_1\} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0], 0) &= [q_0] \\ \delta'([q_0], 1) &= [q_0, q_1] \end{aligned}$$

- $[q_0, q_1]$ を起点に

$$\begin{aligned} \delta(q_0, 0) &= \{q_0\} \\ \delta(q_1, 0) &= \{q_1, q_2\} \\ \delta(q_0, 1) &= \{q_0, q_1\} \\ \delta(q_1, 1) &= \emptyset \end{aligned} \quad \Rightarrow \quad \begin{aligned} \delta'([q_0, q_1], 0) &= [q_0, q_1, q_2] \\ \delta'([q_0, q_1], 1) &= [q_0, q_1] \end{aligned}$$

- $[q_0, q_1, q_2]$ を起点に

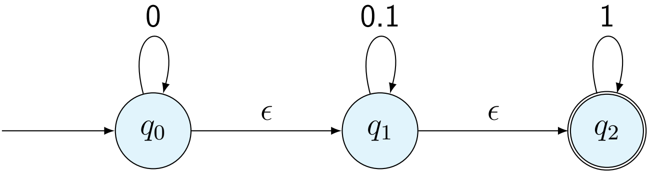
$$
\begin{aligned}
\delta(q_0, 0) &= \{q_0\} \\
\delta(q_1, 0) &= \{q_0, q_2\} \\
\delta(q_2, 0) &= \{q_2\} \\
\delta(q_0, 1) &= \{q_0, q_1\} \\
\delta(q_1, 1) &= \emptyset \\
\delta(q_2, 1) &= \{q_2\}
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\delta'([q_0, q_1, q_2], 0) &= [q_0, q_1, q_2] \\
\delta'([q_0, q_1, q_2], 1) &= [q_0, q_1, q_2]
\end{aligned}
$$

ϵ 動作のある非決定性有限オートマトン

$$M = \langle Q, \Sigma, \delta, q_0, F \rangle \tag{3}$$

- Q : 内部状態の集合
- Σ : 入力アルファベット
- $\delta : Q \times (\Sigma \cup \{\epsilon\}) \to 2^Q$: 状態遷移関数
  文字を読まずに遷移する (ϵ 動作) ことがある
- $q_0 \in Q$: 初期状態
- $F \subseteq Q$: 受理状態

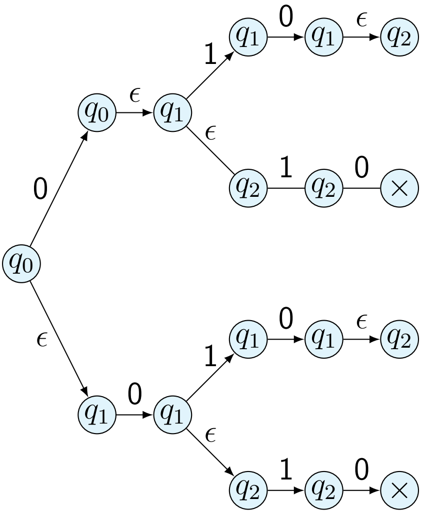
例 1



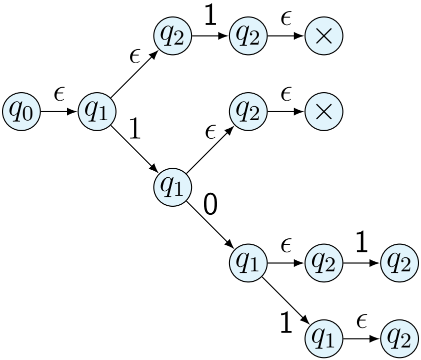
$$Q = \{q_0, q_1, q_2\}, \Sigma = \{0, 1\}, F = \{q_2\}$$

δ	0	1	ϵ
q_0	$\{q_0\}$	$\emptyset$	$\{q_1\}$
q_1	$\{q_1\}$	$\{q_1\}$	$\{q_2\}$
q_2	$\emptyset$	$\{q_2\}$	$\emptyset$

動作例: 入力 010



動作例: 入力 101



q_0
ϵ
q_1
ϵ
q_2
1
q_2
ϵ
×
1
q_1
ϵ
q_2
ϵ
×
0
q_1
ϵ
q_2
1
q_2
1
q_1
ϵ
q_2

ϵ-閉包

- M の状態集合 $Q' \subseteq Q$ の各要素から ϵ-動作のみで到達可能な状態の集合 $\epsilon\text{-CL}(Q')$

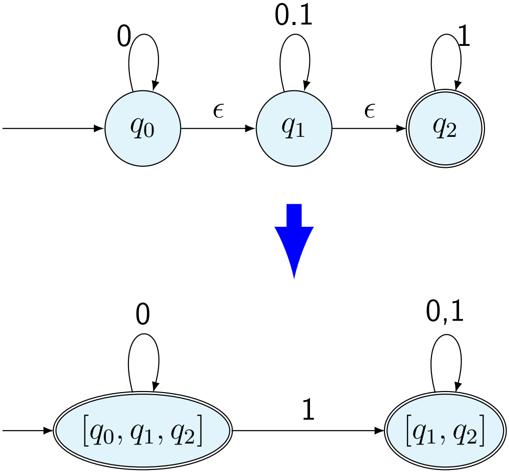
$$
\begin{aligned}
\epsilon\text{-CL}(\{q_0\}) &= \{q_0, q_1, q_2\} \\
\epsilon\text{-CL}(\{q_1\}) &= \{q_1, q_2\} \\
\epsilon\text{-CL}(\{q_2\}) &= \{q_2\}
\end{aligned}
$$

ϵ-NFA $M = \langle Q, \Sigma, \delta, q_0, F \rangle$ に対応した DFAM'

$$M' = \langle Q, \Sigma, \delta', q_0', F' \rangle \tag{4}$$

- $Q' \in 2^Q$
- Σ : 入力アルファベット
- $\delta' : 2^Q \times \Sigma \to 2^Q$: 状態遷移関数
  $\delta' : (Q', a) = \epsilon\text{-CL}\left(\bigcup_{q \in \epsilon\text{-CL}(Q')} \delta(q, a)\right)$
- $q_0' = \epsilon\text{-CL}(\{q_0\})$: 初期状態
- $F' = \left\{A \in 2^Q \mid a \cap F \neq \emptyset\right\}$: 受理状態
- 一文字読んだ遷移後の ϵ 動作を考慮して、遷移先を求める

例 2



- 始状態

$$\epsilon\text{-CL}\left(q_0\right)\left\{q_0, q_1, q_2\right\} \qquad \Rightarrow \qquad \left[q_0, q_1, q_2\right] \text{が始状態}$$

- $\left[q_0, q_1, q_2\right]$ を始点に

$$\begin{aligned}
\delta\left(q_0, 0\right) &= \epsilon\text{-CL}\left(q_0\right) = \left\{q_0, q_1, q_2\right\} \\
\delta\left(q_1, 0\right) &= \epsilon\text{-CL}\left(q_1\right) = \left\{q_1, q_2\right\} \\
\delta\left(q_2, 0\right) &= \emptyset \\
\delta\left(q_0, 1\right) &= \emptyset \\
\delta\left(q_1, 1\right) &= \epsilon\text{-CL}\left(q_1\right) = \left\{q_1, q_2\right\} \\
\delta\left(q_2, 1\right) &= \left\{q_2\right\}
\end{aligned}$$

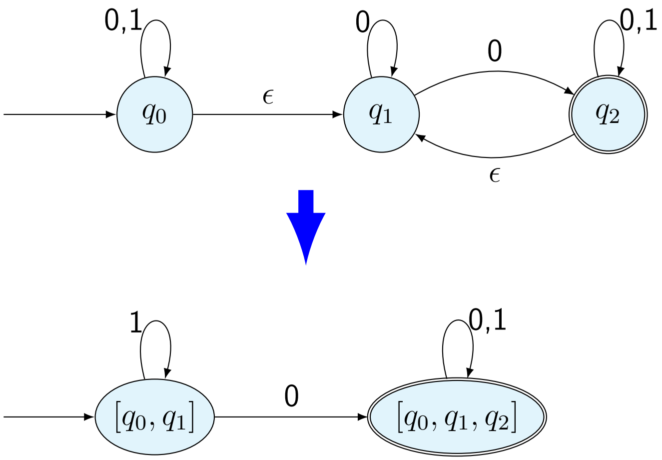
$$\Downarrow$$

$$\begin{aligned}
\delta'\left(\left[q_0, q_1, q_2\right], 0\right) &= \left[q_0, q_1, q_2\right] \\
\delta'\left(\left[q_0, q_1, q_2\right], 1\right) &= \left[q_1, q_2\right]
\end{aligned}$$

- $[q_1, q_2]$ を始点に

$$
\begin{aligned}
\delta(q_1, 0) &= \epsilon\text{-}\mathsf{CL}(q_1) = \{q_1, q_2\} \\
\delta(q_2, 0) &= \emptyset \\
\delta(q_1, 1) &= \epsilon\text{-}\mathsf{CL}(q_1) = \{q_1, q_2\} \\
\delta(q_2, 1) &= \{q_2\} \\
&\Downarrow \\
\delta'([q_1, q_2], 0) &= [q_1, q_2] \\
\delta'([q_1, q_2], 1) &= [q_1, q_2]
\end{aligned}
$$

例 3



- 始状態

$$\epsilon\text{-}\mathsf{CL}\left(q_0\right) = \{q_0, q_1\} \qquad \Rightarrow \qquad [q_0, q_1] \text{ が始状態}$$

- $[q_0, q_1]$ を起点に

$$\begin{aligned}
&\delta\left(q_0, 0\right) = \epsilon\text{-}\mathsf{CL}\left(q_0\right) = \{q_0, q_1\} \\
&\delta\left(q_1, 0\right) = \epsilon\text{-}\mathsf{CL}\left(q_1\right) \cup \epsilon\text{-}\mathsf{CL}\left(q_2\right) = \{q_1, q_2\} \\
&\delta\left(q_0, 1\right) = \epsilon\text{-}\mathsf{CL}\left(q_0\right) = \{q_0, q_1\} \\
&\delta\left(q_1, 1\right) = \emptyset \\
&\qquad\quad \Downarrow \\
&\delta'\left(\left[q_0, q_1\right], 0\right) = \left[q_0, q_1, q_2\right] \\
&\delta'\left(\left[q_0, q_1\right], 1\right) = \left[q_0, q_1\right]
\end{aligned}$$

- $[q_0, q_1, q_2]$ を起点に

$$
\begin{aligned}
\delta(q_0, 0) &= \{q_0, q_1\} \\
\delta(q_1, 0) &= \{q_1, q_2\} \\
\delta(q_2, 0) &= \epsilon\text{-}\mathsf{CL}(q_0) = \{q_1, q_2\} \\
\delta(q_0, 1) &= \{q_0, q_1\} \\
\delta(q_1, 1) &= \emptyset \\
\delta(q_2, 0) &= \epsilon\text{-}\mathsf{CL}(q_2) = \{q_1, q_2\} \\
&\Downarrow \\
\delta'([q_0, q_1, q_2], 0) &= [q_0, q_1, q_2] \\
\delta'([q_0, q_1, q_2], 1) &= [q_0, q_1, q_2]
\end{aligned}
$$

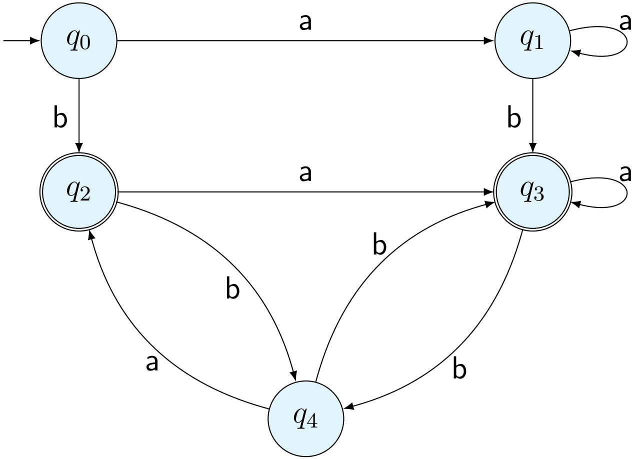
DFA の簡素化

- 同じ文字列を受理する DFA のうちで、状態数の最小の DFA への変換
- 状態の集合から入力による遷移先の集合に注目する

最小化アルゴリズム

状態を受理状態の集合と、それ以外の状態の集合に分ける
repeat
　各状態集合に対して、各入力による遷移先が同じ集合に分割する
until 新たな状態集合はない
元の初期状態を含む状態集合を新たな初期状態に、元の受理状態のみを含む状態集合を新たな受理集合に

例 1



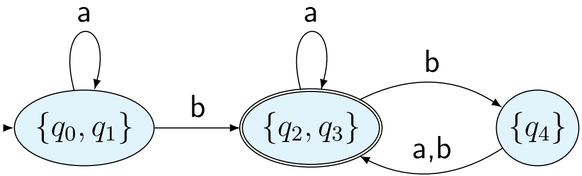
- $\{q_2, q_3\}$

$$\{q_2, q_3\} \xrightarrow{\mathsf{a}} \{q_3\} \subset \{q_2, q_3\}$$
$$\{q_2, q_3\} \xrightarrow{\mathsf{b}} \{q_4\} \subset \{q_0, q_1, q_4\}$$

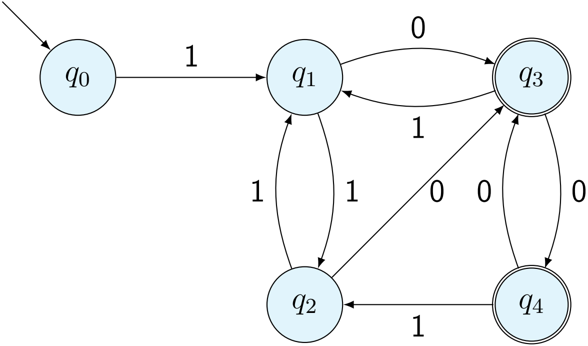
  - 分割の必要なし
- $\{q_0, q_1, q_4\}$

$$\{q_0, q_1\} \xrightarrow{\mathsf{a}} \{q_1\} \subset \{q_0, q_1\}$$
$$\{q_4\} \xrightarrow{\mathsf{a}} \{q_2\} \subset \{q_2, q_3\}$$
$$\{q_0, q_1\} \xrightarrow{\mathsf{b}} \{q_2, q_3\}$$
$$\{q_4\} \xrightarrow{\mathsf{b}} \{q_3\} \subset \{q_2, q_3\}$$

  - $\{q_0, q_1\}$ と $\{q_4\}$ の二つに分割
- 再度全ての状態集合に確認し、新たな分割は無い


a
a
b
b
a,b
$\{q_0, q_1\}$
$\{q_2, q_3\}$
$\{q_4\}$

例 2



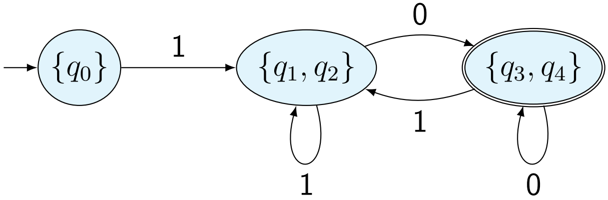
- $\{q_0, q_1, q_2\}$

$$\{q_0\} \xrightarrow{1} \{q_1\} \subset \{q_1, q_2\}$$
$$\{q_1, q_2\} \xrightarrow{0} \{q_3\} \subset \{q_3, q_4\}$$
$$\{q_1, q_2\} \xrightarrow{1} \{q_1, q_2\}$$

  - $\{q_1\}$ と $\{q_1, q_2\}$ に分割
- $\{q_3, q_4\}$

$$\{q_3, q_4\} \xrightarrow{0} \{q_3, q_4\}$$
$$\{q_3, q_4\} \xrightarrow{1} \{q_1, q_2\}$$

  - 分割の必要なし


{q0}
1
{q1, q2}
0
1
{q3, q4}
1
0